OPTIMUM DESIGN OF FIBRE-REINFORCED COMPOSITE STRUCTURAL ELEMENTS

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By

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to the

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TO MY

Reverend Mother & Father

CERTIFICATE

This is to certify that the thesis entitled "OPTIMUM DESIGN OF FIBRE-REINFORCED COMPOSITE STRUCTURAL ELEMENTS" has been carried out under our joint supervision and has not been submitted elsewhere for a degree.

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NOMENCLATURE

a	-	plate length
A_{ij}	-	material stiffnesses
A_{mn}, B_{mn}, C_{mn}	-	coefficients of Fourier expansion
$[A]$	-	material stiffness matrix
$[A^*]$	-	inverse of $[A]$
b	-	plate width
B_{ij}	-	coupling matrix elements
B^*_{ij}	-	elements of $[B^*]$
$[B]$	-	stiffness coupling matrix
$[B^*]$	-	matrix formed by $[A]^{-1}[B]$
$[C^*]$	-	matrix formed by $[B][A]^{-1}$
D_{ij}	-	flexular rigidities
D^*_{ij}	-	reduced flexular rigidities
$[D]$	-	flexular stiffness matrix
$[D^*]$	-	reduced stiffness matrix
E_i	-	Young's modulus in ith direction
f_1, f_2	-	arbitrary functions of integration
F	-	objective function
g_j	-	jth constraint function
G_{LT}	-	shear modulus
$\overline{G}_q$	-	gradient vector at point $\overline{X}_q$

h	-	thickness of plate
H_q	-	symmetric, positive definite matrix
J	-	Hessian matrix
$\bar{K}_{ij}$	-	elements of $[\bar{K}]$
$[K]$	-	stiffness matrix referred to natural axes
$[\bar{K}]$	-	stiffness matrix referred to arbitrary axes
l	-	beam length
m, n	-	number of nodal lines in x and y directions, respectively
m, p, k, s	-	upper limit for subscripts
M	-	magnitude of stress couple per unit length
M_q, N_q	-	up-dating matrices for H_q
$\{M\}$	-	stress couples per unit length
N, N_j	-	half number of plies, number of design variables
N_x	-	stress resultant per unit length in x - direction
$\{N\}$	-	stress resultant per unit length
p_{mn}	-	natural frequency
P	-	magnitude of a force
q	-	normal pressure
q_{mn}	-	harmonic function of time
r, r_i	-	penalty parameter
R	-	aspect ratio, a/b
S	-	upper limit on shear stresses
$\bar{S}_q$	-	search direction in qth minimization

$[S]$ - compliance matrix

t - time

t_i - thickness of lamina

u, v, w, w_i - mid-plane displacements

u^o, v^o - reference surface displacements

U - Airy stress function

W_j - weight of the plate

W_{mn} - plate modal eigen function

x_i - components of design vector

x, y, z - rectangular coordinates

$\bar{X}_q$ - design vector at qth iteration

X, Y - upper limits on lamina normal stresses

Y_{mn} - trace of W_{mn} in plane x = constant

$\bar{Y}_q$ - vector notation for $\bar{G}_{q+1} - \bar{G}_q$

α, α_j - lower bound on laminate critical buckling load

α_q, α^*_q - step length in qth minimization

β, β_j - lower bound on laminate fundamental frequency

δ - relative difference in function values in successive minima

$\bar{\Delta}$ - vector defined by $\bar{X}_m (r_{i-1}) - \bar{X}_m (r_i)$

ε	-	limit for termination of optimization process
ε_i, ε_{ij}	-	strain components
$\{\varepsilon_o\}$	-	reference surface strains
θ, θ_i	-	ply orientation angle
μ	-	Poisson's Ratio for isotropic material
ν_{LT}	-	major Poissons ratio of ply
ξ	-	integration variable
ρ^*	-	average mass density
σ_i, σ_x, σ_y	-	normal stresses
σ_{ij}	-	stress components
τ_{xy}	-	shear stress
ϕ	-	penalty function
ϕ_{mn}, ψ_{mn}	-	roots of frequency equation
χ	-	reference surface curvatures

Subscripts :

cr	-	critical value
i	-	array index
i, j, k, l, m, n	-	tensor subscripts
l	-	value at distance l from origin
L, T	-	material principal axes
max	-	maximum value

o	-	origin
opt	-	optimum value
q	-	iteration in minimization

Superscripts

k	-	kth lamina
T	-	transpose of a matrix

SYNOPSIS

Continuous fibre-reinforced composite materials are an important class of modern materials. Their high strength to weight and modulus to weight ratios, controlled anisotropy and formability make them superior to many conventional materials for various applications. The strength, elastic modulus and many other properties of such materials are highly direction dependent. The present studies were carried out to show the feasibility of obtaining theoretical optimum parameters of laminated composite beams and plates subjected to multiple behaviour constraints. A number of specific optimization problems were solved using the penalty function method of optimization for constrained problems and the Davidon - Fletcher - Powell method for solving the sequence of unconstrained minimization problems. Tests were performed on the fibre-reinforced composite plates at different orientation angles to find the natural frequencies of vibration for simply supported boundary edge conditions.

CHAPTER I

INTRODUCTION

The need for high strength to weight ratio materials has increased considerably with the advance of technology. Specially in structures where weight is a problem, e.g. space-crafts, aircrafts etc., the high specific strength materials are desirable. Since the directional properties can be controlled very precisely in composite materials, they have a higher specific strength as compared to the popular isotopic materials. Since the structures are in general subjected to vibrations, in-plane loading, transverse loading etc., the studies have been conducted to find an optimum solution with different combinations of the above constraints.

1.1 LITERATURE SURVEY

The study of optimal design of fibre reinforced composites (FRC) has started recently with increased application of high performance F.R.C materials for efficient design. The minimum weight design of laminates for strength and membrane stiffness was studied extensively by Foye [9]. Multiple in-plane loading conditions were considered and a random search method

was used to find the ply orientation angles such that the strength and stiffness requirements would be satisfied with the smallest number of plies. Another procedure for the optimum design of laminates has been reported by Waddoups [10]. Minimum weight designs are obtained considering strength constraints under multiple distinct loading conditions. Either the Hill-Tsai or the minimum strain criterion of failure may be used and all lamina are assumed to behave linearly to failure. The search procedure employed is a systematic 'try them all' procedure which is quite exhaustive. Both of these studies deal directly with discrete number of plies and they also treat ply orientations as design variables. Waddoups and Co-workers [11] have reported a structural synthesis capability for a class of anisotropic plate structures. Kicher and Chao [12] have reported the development of a structural optimisation capability for stiffened fibre-composite cylinders. Multiple load conditions are considered and strength as well as buckling failure modes are guarded against. The design veriables include stiffener dimensions and spacing, fibre volume content and ply orientation angles. Schmit and Farshi [8] have developed a laminate optimisation capability in which the thicknesses of material at specified orientation angles

are treated as the only design variables. That is to say, the material to be used would be selected in advance from a discrete set of available well-characterized systems and the orientation angles to be used would also be preassigned. By treating the thickness of material at each preassigned orientation as a continuous design variable the difficulties associated with discrete integer design variables are avoided.

1.2 PRESENT WORK

In this work, the optimum design of symmetric laminates is considered by taking the ply thicknesses (t_i) / the number of plies (N) as design variables so as to satisfy certain specified constraints on the natural frequencies, buckling load and deflection.

An experimental study has also been made to find the dynamic response of the laminates at different orientations since the closed form solutions for different orientations are not available.

CHAPTER II

ANALYSIS OF LAMINATED BEAMS AND PLATES

A laminated beam or plate is composed of an arbitrary number of bonded plies, each of which may be inhomogeneous having different thickness and orthotropic elastic properties. As a mathematical approximation, the laminate is considered to be an in-plane homogeneous and transversely heterogeneous continuum. The transverse heterogeneity has a step-wise variation in material properties between plies.

The analysis of deflection of a simply supported beam under uniformly distributed load conditions is done within the frame work of the small deflection theory of thin plates and based on the following assumptions:

1. The material homogeneity through the plate thickness is there, which means that the gross laminate properties will apply.
2. If the Saint Venant's principle is imposed, the plate length 'l' must be much greater than the depth '2b'. Also, it will be assumed that the plate thickness 'h' is unity, much smaller than '2b'.

Thus the analysis is, theoretically, applicable only for a long, thin, homogeneous anisotropic plate.

The analysis of dynamic behaviour of the laminated plates is also done within the frame work of the small deflection theory of thin plates and based on the following assumptions:

1. Each layer is homogeneous, linearly elastic and under a state of plane stress.
2. The plate thickness 'h' is small in relation to the lateral dimensions, and the transverse-deflection w is small compared to h.
3. Plane, normal cross-section before deformation remains plane and normal to the reference surface of the plate during deformation; thus transverse-shear deformation can be neglected.

Under the above assumptions, the problem of the transverse vibration of the laminated plates without in-plane forces becomes that of a homogeneous plate once a constitutive equation for the laminated plate is obtained.

2.1 LAMINATED PLATES

In this section we describe the general notations and constitutive equations for the laminated plates with appropriate simplifications for using in further analysis.

2.1.1 General Laminated Plates

A laminated plate consists of 2N constituent plies of different orthotropic material and thickness with arbitrary ply orientations. The structure axes of the laminate are denoted by x, y, and z, and for each ply the two symmetric axes of material in the plane of the ply by L and T. The ply orientation angle θ is the angle between the x - and L - axes, and measured in the counter-clock-wise direction from the positive x - axis as shown in figure 1. The structure coordinate system is located so that the x - y plane (z = 0) coincides with the middle surface of the laminate. The plies are numbered from 1 to 2N from the bottom ply to the top ply with the bottom surface of the ith ply located at the distance z_i from the middle surface as shown in figure 2. The ply orientation of the laminate will be denoted by $(\theta_1, \theta_2 \ldots)$.

For orthotropic plies under a state of plane stress, four independent elastic constants are required to completely characterize the elastic behaviour.

These principal elastic constants are the Young's moduli in the two principal directions E_L and E_T; shear modulus, G_{LT}; and the major Poisson's ratio, ν_{LT}. The constitutive equation, which characterizes the structural behaviour of the laminate, can be obtained once the principal elastic constants, thickness and the ply orientation of each unit ply are known.

The constitutive equation is the relations between stress resultants $\{N\}$ and stress couples $\{M\}$ and the reference surface strains $\{\varepsilon_o\}$ and curvatures $\{\chi\}$. By defining the stress resultants and stress couples according to the linear theory of plates:

$$N = \int_{-h/2}^{h/2} \sigma \, dz \qquad M = \int_{-h/2}^{h/2} \sigma \, z \, dz$$

the constitutive equation for the laminate can be written in the following matrix form :

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ \hline M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{13} & B_{11} & B_{12} & B_{13} \\ A_{12} & A_{22} & A_{23} & B_{12} & B_{22} & B_{23} \\ A_{13} & A_{23} & A_{33} & B_{13} & B_{23} & B_{33} \\ \hline B_{11} & B_{12} & B_{13} & D_{11} & D_{12} & D_{13} \\ B_{12} & B_{22} & B_{23} & D_{12} & D_{22} & D_{23} \\ B_{13} & B_{23} & B_{33} & D_{13} & D_{23} & D_{33} \end{array}\right] \begin{Bmatrix} \varepsilon_{oz} \\ \varepsilon_{oy} \\ \varepsilon_{oxy} \\ \hline \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} \quad (1)$$

where, $[A]$ is the in-plane membrane stiffness matrix, lb./in.

$[B]$ is the stiffness coupling matrix, lb.

and $[D]$ is the flexural stiffness matrix, lb.-in.

Since the reference surface (z = 0) is taken to coincide with the middle surface, the stress couples are computed with regard to the middle surface, and the reference surface strains are the middle surface strains. The reference surface could be chosen to be other than the middle surface. Consideration of $[B]$ matrix in the constitutive equation indicates the coupling phenomena between stress resultants and the curvatures and between the stress couples and the middle surface strains which do not occur in the theory of homogeneous plates.

It is convenient to express M and ε_o in terms of χ and N, for the bending analysis of laminated plates. Solving the first three equations in (1) for ε_o, in terms of χ and N, and introducing the results into the remaining three equations yields,

$$\varepsilon_o = [A^*] N + [B^*]$$

$$M = [C^*] N + [D^*]$$

where $[A^*] = [A]^{-1}$

$$[B^*] = - [A]^{-1} [B]$$

$$[C^*] = [B] [A]^{-1}$$

and $[D^*] = [D] - [B] [A]^{-1} [B]$

It is noted that while $[A^*]$ and $[D^*]$ are symmetric matrices, the same need not be true for $[B^*]$ and $[C^*]$. If the laminated plate is free from the stress resultants (no external in-plane force), the stress couples and the middle surface strains become,

$$\begin{aligned} M &= [D^*]\chi \\ \varepsilon_o &= [B^*]\chi \end{aligned} \tag{2}$$

The matrix $[D^*]$ is called the reduced stiffness matrix.

2.1.2 Special Laminated Plates

For the laminated plate having an elastic symmetry about the middle surface, the coupling stiffness matrix is a null matrix, and the bending and membrane behaviours are uncoupled. The reduced stiffness matrix is, therefore identical to the flexural stiffness matrix, and the middle surface strains due to bending vanish. For the purpose of this study, the laminate is considered to be orthotropic if the laminate is uncoupled and its membrane and flexural stiffness matrices are orthotropic, i.e. $A_{13} = A_{23} = D_{13} = D_{23} = 0$. An orthotropic laminate has three planes of elastic symmetry: the middle surface and the two planes whose normals are along the x and y axes.

For coupled laminates, the sufficient condition that the reduced stiffness matrix is orthotropic ($D^*_{13} = D^*_{23} = 0$) is

(a) $$A_{13} = A_{23} = 0$$
$$D_{13} = D_{23} = 0$$
$$B_{13} = B_{23} = 0$$

or

(b) $$A_{13} = A_{23} = 0$$
$$D_{13} = D_{23} = 0$$
$$B_{11} = B_{12} = B_{22} = B_{33} = 0$$

The elements B^*_{13}, B^*_{23}, B^*_{31}, B^*_{32} vanish also for the condition (a).

For the laminated plate consisting of constituent plies of the same orthotropic material and thickness such as a laminated fibrous composite plate, the constituent equation becomes only function of the ply orientations once the principal elastic constants and thickness of ply materials are specified. Depending upon the manner in which the individual plies are oriented with respect to each other and arranged about the middle surface of the laminate, some of the elements of the membrane, coupling and flexural stiffness matrices vanish as follows:

1. If the plies are oriented symmetrically about the middle surface, $(\theta_1, \theta_2, \theta_3, \theta_2, \theta_1)$, then

$$\text{All } B_{ij} = 0$$

2. If the plates are oriented anti-symmetrically about the middle surface, $(\theta_1, \theta_2, \theta_3, -\theta_3, -\theta_2, -\theta_1)$, then

$$A_{13} = A_{23} = 0$$

$$D_{13} = D_{23} = 0$$

$$B_{11} = B_{12} = B_{22} = B_{33} = 0$$

3. If some of the plies are oriented symmetrically and the others are oriented anti-symmetrically about the middle surface $(\theta_1, \theta_2, \theta_3, -\theta_2, \theta_1)$, then

$$B_{11} = B_{12} = B_{22} = B_{33} = 0$$

4. If some of the plies are oriented symmetrically about the middle surface and the others are oriented either at 0° or 90° $(\theta_1, 0^\circ, \theta_3, 90^\circ, \theta_1)$, then

$$B_{13} = B_{23} = 0$$

5. If the plies are oriented at either 0° or 90° regardless of symmetry about the middle surface, then

$$A_{13} = A_{23} = 0$$

$$D_{13} = D_{23} = 0$$

$$B_{13} = B_{23} = 0$$

When the laminate has isotropic plies, such as glue layers in fiberous composite plates, in addition to the orthotropic plies as its constituents, case (5) is uneffected, and case (1) to case (4) are also uneffected provided the isotropic plies possess identical thickness and elastic properties, and are located symmetrically about the middle surface.

2.2 ANALYSIS OF LAMINATED BEAMS

In this section we derive the deflection components u and v in x - and y - directions for a simply supported beam under uniformly distributed load (see figure 3). The approach is same as the one advanced by Hashin [13] for anisotropic beams in a state of plane stress.

2.2.1 Boundary Conditions

The boundary conditions to be satisfied by the stresses and loading are as follows:

$$\begin{aligned}
\tau_{xy}(x, -b) &= \sigma_y(x, -b) = 0 \\
\tau_{xy}(x, b) &= 0 \\
\sigma_y(x, b) &= q \\
M_o &= 0 \\
M_i &= 0
\end{aligned} \qquad (3a)$$

$$P_{1o} = 0$$
$$P_{1l} = 0$$
$$P_{2o} = -ql/2$$
$$P_{2l} = ql/2 \qquad (3b)$$

where σ_x and σ_y are normal stresses in the x - and y - directions respectively, τ_{xy} is the shear stress in the xy plane, P and M represent the force and the moment, respectively, 1 and 2 representing x - and y - directions and o and l representing values at origin and at length l.

2.2.2 Stress Function

In the absence of body forces, the plane equations of equilibrium will be identically satisfied by the Airy stress function U (x, y) defined by

$$\sigma_x = \frac{\partial^2 U}{\partial y^2}$$
$$\sigma_y = \frac{\partial^2 U}{\partial x^2} \qquad (4)$$
$$\tau_{xy} = -\frac{\partial^2 U}{\partial x\, \partial y}$$

and the strains ε_{ij} are given by

$$\varepsilon_{ij} = S_{ijkl} \quad \sigma_{kl} \qquad (5)$$

where S_{ijkl}, the compliance matrix, is the inverse of the stiffness matrix, K_{ij}, when S_{ijkl} is reduced to variables with two subscripts as shown on page 15.

The only surviving compatibility equation is

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \, \partial y} \tag{6}$$

substituting equations (4) and (5) into equation (6), the differential equation for U can be obtained as

$$S_{2222} \, U_{,xxxx} - 4 \, S_{2212} \, U_{,xxxy} + 2 \, (S_{1122} + 2 \, S_{1212}) \, U_{,xxyy}$$

$$- 4 \, S_{1112} \, U_{,xyyy} + S_{1111} \, U_{,yyyy} = 0 \tag{7}$$

where partial differentiation is denoted by a comma.

Using equations (3(a)) and (3(b)) the boundary conditions for the stress function can be stated as [13]

$$U\,(x,\ -b) = \int_0^x (x - \xi)\, \sigma_y\,(\xi,\ -b)\, d\xi = 0 \qquad 8(a)$$

$$U\,(x,\ -b)_{,x} = \int_0^x \sigma_y\,(\xi,\ -b)\, d\xi = 0 \qquad 8(b)$$

$$U\,(x,\ -b)_{,y} = - \int_0^x \tau_{xy}\,(\xi,\ -b)\, d\xi = 0 \qquad 8(c)$$

$$\begin{aligned} U\,(x,\ b) = & \int_0^l (x - \xi)\, \sigma_y\,(\xi,\ -b)\, d\xi \\ & -2\,b \int_0^l \tau_{xy}\,(\xi,\ -b)\, d\xi + M_1 \\ & + P_{21}\,(l - x) + P_{11}\, b \\ & + \int_x^l (\xi - x)\, \sigma_y\,(\xi,\ b)\, d\xi \end{aligned}$$

$$\text{or } U\,(x,\ b) = \frac{ql}{2}\,(2l - b - 3x - x^2/l) \qquad 8(d)$$

$$U(x, b)_{,y} = -\int_0^l \tau_{xy}(\xi, -b)\, d\xi + P_{11}$$

$$+ \int_x^l \tau_{xy}(\xi, b)\, d\xi = -ql/2 \qquad 8(e)$$

In the following discussions, to avoid complicacy, the following notation will be used for the compliance matrix elements:

$$S_{1111} = S_{11},\ S_{2222} = S_{22},\ S_{1112} = S_{13}$$

$$S_{1122} = S_{12},\ S_{1212} = S_{33}$$

The other elements can be defined in a similar manner.

Let us assume the stress-function to be defined as follows:

$$U(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{M+3} C_{mn}\, x^m y^n$$

such that $m + n \leqslant M + 3$ and M is the highest power of x in equations (8(a)) to (8(e))

$$\therefore\ M = 2 \text{ and } m + n \leqslant 2 + 3 = 5.$$

$$\begin{aligned}\therefore\ U(x, y) = {} & C_{00} + C_{01}\, y + C_{02}\, y^2 + C_{03}\, y^3 + C_{04}\, y^4 \\ & + C_{05}\, y^5 + C_{10}\, x + C_{11}\, xy + C_{12}\, xy^2 \\ & + C_{13}\, xy^3 + C_{14}\, xy^4 + C_{20}\, x^2 \\ & + C_{21}\, x^2y + C_{22}\, x^2y^2 + C_{23}\, x^2y^3 \qquad (9)\end{aligned}$$

Using compatibility equations, stress-strain relations and stress-function, we get the recursion relations for C_{mn} as

$$S_{22}\ (m+2)\ (m+1)\ m\ (m-1)\ C_{m+2,n-2} - 4\ S_{23}\ (m+1)\ m$$

$$* (m-1)\ (n-1)\ C_{m+1,n-1} + 2\ (S_{12} + 2\ S_{33})\ m\ (m-1)\ n$$

$$* (n-1)\ C_{mn} - 4\ S_{13}\ (m-1)\ (n+1)\ n\ (n-1)\ C_{m-1,n+1}$$

$$+ S_{11}\ (n+2)(n+1)\ n\ (n-1)\ C_{m-2,n+2} = 0 \qquad (10)$$

when $m \geqslant 2$ and $n \geqslant 2$

The total number of coefficients C_{mn} is given by

$$S = \frac{1}{2}\ (m+1)\ (M+8) = \frac{1}{2} \times 3 \times 10 = 15$$

and total number of recursion relations is given by

$$R = \frac{1}{2}\ M\ (M+1) = \frac{1}{2} \times 2\ (2+1) = 3$$

$\therefore$ For $m = 2$, $n = 2$ we have from equation (10)

$$(S_{12} + 2\ S_{33})\ C_{22} + 3\ S_{11}\ C_{04} - 3\ S_{13}\ C_{13} = 0 \qquad (11)$$

For $m = 2$, $n = 3$ we have from equation (10)

$$(S_{12} + 2\ S_{33})\ C_{23} - 4\ S_{13}\ C_{14} + 5\ S_{11}\ C_{05} = 0 \qquad (12)$$

For $m = 3$, $n = 2$ we have from equation (10)

$$-2\ S_{13}\ C_{23} + S_{11}\ C_{14} = 0 \qquad (13)$$

By evaluating the partial derivative of U (x, y) with respect to y at y = b and equating coefficients of like powers of x in this equation and equation (8(e)) we get

$$C_{01} + 2b\, C_{02} + 3b^2\, C_{03} + 4\, C_{04}\, b^3 + 5\, C_{05}\, b^4 = -\, ql/2 \tag{14}$$

$$C_{11} + 2b\, C_{12} + 3b^2\, C_{13} + 4b^3\, C_{14} = 0 \tag{15}$$

$$C_{21} + 2b\, C_{22} + 3b^2\, C_{23} = 0 \tag{16}$$

Again evaluating the partial derivative of U (x, y) with respect to y at y = -b and equating coefficients of like powers of x in this equation and equation (8(c)) we get

$$C_{01} - 2b\, C_{02} + 3b^2\, C_{03} - 4b^3\, C_{04} + 5b^4\, C_{05} = 0 \tag{17}$$

$$C_{11} - 2b\, C_{12} + 3b^2\, C_{13} - 4b^3\, C_{14} = 0 \tag{18}$$

$$C_{21} - 2b\, C_{22} + 3b^2\, C_{23} = 0 \tag{19}$$

Now evaluating U (x, y) at y = b and equating coefficients of like powers of x with those in equation (8(d)), we get

$$C_{00} + b\, C_{01} + b^2\, C_{02} + b^3\, C_{03} + b^4\, C_{04} + b^5\, C_{05} = \frac{ql}{2}(2l-b) \tag{20}$$

$$C_{10} + b\, C_{11} + b^2\, C_{12} + b^3\, C_{13} + b^4\, C_{14} = -\, 3\, ql/2 \tag{21}$$

$$C_{20} + b\, C_{21} + b^2\, C_{22} + b^3\, C_{23} = -\, q/2 \tag{22}$$

Similarly evaluating U (x, y) at y = -b and

equating coefficients of like powers of x with those in equation (8(a)), we get

$$C_{00} - b\, C_{01} + b^2\, C_{02} - b^3\, C_{03} + b^4\, C_{04} - b^5\, C_{05} = 0 \qquad (23)$$

$$C_{10} - b\, C_{11} + b^2\, C_{12} - b^3\, C_{13} + b^4\, C_{14} = 0 \qquad (24)$$

$$C_{20} - b\, C_{21} + b^2\, C_{22} - b^3\, C_{23} = 0 \qquad (25)$$

A simultaneous solution of equations (11) to (25) gives all the 15 coefficients, C_{mn}, to completely define the stress-function $U\ (x, y)$. The coefficients C_{mn} are given below:

$$C_{00} = ql\ ((4l - b)\ S_{11} + 3\ S_{13}\ b)\ /\ 8\ S_{11} \qquad (26)$$

$$C_{01} = -\ ql\ (3l + b)/4b + qb\ (8\ S_{13}^2 - (S_{12} + S_{33})S_{11})/40\ S_{11}^2 \qquad (27)$$

$$C_{02} = -\ ql/8b - 3\ S_{13}\ ql\ /\ 4\ S_{11}\ b \qquad (28)$$

$$C_{03} = ql^2\ /\ 4b^3 - q\ (8\ S_{13}^2 - (S_{12} + 2\ S_{33})\ S_{11})\ /\ 20\ b\ S_{11}^2 \qquad (29)$$

$$C_{04} = 3\ S_{13}\ ql\ /\ 8\ S_{11}\ b^3 \qquad (30)$$

$$C_{05} = q\ (8\ S_{13}^2 - (S_{12} + 2\ S_{33})\ S_{11})\ /\ 40\ S_{11}^2\ b^3 \qquad (31)$$

$$C_{10} = -\ 3\ ql\ /\ 4 - S_{13}\ qb\ /\ 4\ S_{11} \qquad (32)$$

$$C_{11} = 9\ ql\ /\ 8b \qquad (33)$$

$$C_{12} = -\ S_{13}\ q\ /\ 2\ S_{11}\ b \qquad (34)$$

$$C_{13} = 3\ ql\ /\ 8\ b^3 \qquad (35)$$

$$C_{14} = S_{13}\ q\ /\ 4\ S_{11}\ b^3 \qquad (36)$$

$$C_{20} = -q\ /\ 4 \qquad (37)$$

$$C_{21} = -\ 3q\ /\ 8b \qquad (38)$$

$$C_{22} = 0 \qquad (39)$$

$$C_{23} = q\ /\ 8\ b^3 \qquad (40)$$

2.2.3 Deflection Equations and the Stress Components

The deflections are given by

$$u = S_{11} \int u_{,yy}\, dx + S_{12} U_{,x} - 2 S_{13} U_{,y} + f_1 (y) \tag{41}$$

and $$v = S_{12} U_{,y} + S_{22} \int u_{,xx}\, dy - 2 S_{23} U_{,x} + f_2(x) \tag{42}$$

The arbitrary functions $f_1 (y)$ and $f_2 (x)$ are given by using

$$\varepsilon_{xy} = \frac{1}{2} \gamma_{xy} = \frac{1}{2} (u_{,y} + v_{,x}) = S_{13} U_{,yy} + S_{23} U_{,xx} - 2 S_{33} U_{,xy} \tag{43}$$

and three boundary conditions, namely $u = 0$, $v = 0$ at $x = 0$, $y = 0$ and $v = 0$ at $x = l$, $y = 0$.

Finally, we get the deflection equations as

$$\begin{aligned} u (x, y) = {} & 2 (S_{11} C_{02} - S_{13} C_{11} + S_{12} C_{20})\, x \\ & + (4 (S_{13} C_{02} - S_{33} C_{11} + S_{23} C_{20}) \\ & - (3 S_{11} C_{03} - 4 S_{13} C_{12} + (S_{12} + 4 S_{33}) C_{21})\, l \\ & - S_{11} (C_{13}\, l^2 + C_{23}\, l^3))\, y + (S_{11} C_{12} - 2 S_{13} C_{21}) x^2 \\ & + 2 (3 S_{11} C_{03} - 2 S_{13} C_{12} + S_{12} C_{21})\, xy \\ & + (S_{12} C_{12} + 2 (3 S_{13} C_{03} - (S_{12} + 2 S_{33}) C_{12} \\ & + 2 S_{23} C_{21}))\, y^2 + 3 S_{11} C_{13}\, x^2 y \\ & + 6 (2 S_{11} C_{04} - S_{13} C_{13})\, xy^2 + (8 S_{13} C_{04} - S_{12} C_{13} \\ & - 4 S_{33} C_{13})\, y^3 + 2 S_{11} C_{23}\, x^3 y + 6(S_{11} C_{14} \\ & - S_{13} C_{23})\, x^2 y^2 + 2 (10 S_{11} C_{05} - 4 S_{13} C_{14} \\ & + S_{12} C_{23})\, xy^3 \end{aligned} \tag{44}$$

and

$$
\begin{aligned}
v(x, y) = & ((3 S_{11} C_{03} - 4 S_{13} C_{12} + (S_{12} + 4 S_{33}) C_{21}) 1 \\
& + S_{11} (C_{13} + C_{23} 1) 1^2) x \\
& + 2 (S_{12} C_{02} - S_{23} C_{11} + S_{22} C_{20}) y \\
& - (3 S_{11} C_{03} + (S_{12} + 4 S_{33}) C_{21} + 4 S_{13} C_{12}) x^2 \\
& + 2 (S_{12} C_{12} - 2 S_{23} C_{21}) xy + (3 S_{12} C_{03} \\
& - 2 S_{23} C_{12} + S_{22} C_{21}) y^2 - S_{11} C_{13} x^3 \\
& + 3 S_{12} C_{13} xy^2 + 2 (2 S_{12} C_{04} - S_{23} C_{13}) y^3 \\
& - S_{11} C_{23} x^4 + 3 S_{12} C_{23} x^2 y^2 \\
& + 4 (S_{12} C_{14} - S_{23} C_{23}) xy^3 + (5 S_{12} C_{05} \\
& - S_{23} C_{14} + \tfrac{1}{2} S_{22} C_{23}) y^4
\end{aligned}
\tag{45}
$$

The mid-plane displacements are given by

$$
\begin{aligned}
u(x, 0) = & 2 (S_{11} C_{02} - S_{13} C_{11} + S_{12} C_{20}) x \\
& + (S_{11} C_{12} - 2 S_{13} C_{21}) x^2
\end{aligned}
$$

and

$$
\begin{aligned}
v(x, 0) = & (3 S_{11} C_{03} - 4 S_{13} C_{12} + 1 S_{11} C_{13} \\
& + (S_{12} + 4 S_{33}) C_{21} + 1^2 S_{11} C_{23}) 1x \\
& - (3 S_{11} C_{03} - 4 S_{13} C_{12} + (S_{12} + 4 S_{33}) C_{21}) x^2 \\
& - S_{11} C_{13} x^3 - S_{11} C_{23} x^4
\end{aligned}
\tag{46}
$$

The stress components for a simply supported beam under uniformly distributed load as given in reference [7] are (after transforming to the coordinate system shown in figure 3):

$$\sigma_x^{(k)} = \bar{K}_{11}^{(k)} \; \underline{A} \; \underline{B} \; M_x(x)$$

$$\sigma_z^{(k)} = \bar{K}_{12}^{(k)} \; \underline{A} \; \underline{B} \; M_x(x) \tag{47}$$

and

$$\tau_{xy}^{(k)} = \bar{K}_{14}^{(k)} \; \underline{A} \; \underline{B} \; M_x(x)$$

where

$$\underline{A} = \frac{A_{11}}{B_{11}^2 - A_{11} D_{11}} \tag{47(a)}$$

$$\underline{B} = \frac{B_{11}}{A_{11}} + y \; , \tag{47(b)}$$

$\bar{K}_{ij}$ represents the stiffness at any angle θ and the superscript k indicates the k^{th} ply.

2.3 ANALYSIS OF LAMINATED PLATES

In this section we derive the expressions for the transverse deflection, the critical buckling load and the frequency of natural vibration for a simply supported laminated orthotropic plate within the frame work of the assumptions made in section 2.1, and at the beginning of this chapter.

2.3.1 Transverse Deflection Analysis of Laminated Plates

The classical laminated plate theory [1, 2, 3] reveals a coupling phenomenon between bending and middle plane extensions which was not found in the linear theory of homogeneous plates. This phenomenon occurs through the constitutive relations given in equation (1). Due to limited existence of solutions to boundary value problems, the effect of coupling on the mechanical behaviour of composites is not well understood. In this section the effect of coupling on the response of simply supported rectangular laminates of special construction under transverse load is studied [4] .

Closed form solutions can be obtained for coupled laminated plates in which $[A]$ and $[D]$ matrices of equation (1) are orthotropic (i.e. $A_{13} = A_{23} = D_{13} = D_{23} = 0$) and either B_{11}, B_{22} or B_{13} and B_{23}

are the only non-zero elements of the $[B]$ matrix. Using a standard x, y, z coordinate system, the governing equations for unbalanced cross-ply and angle ply plates in terms of displacements can be expressed as [3]:

$$A_{11} u^{o}_{,xx} + A_{33} u^{o}_{,yy} + (A_{12} + A_{33}) v^{o}_{,xy} - B_{11} w_{,xxx} - 3 B_{13} w_{,xxy} - B_{23} w_{,yyy} = 0 \tag{48}$$

$$(A_{12} + A_{33}) u^{o}_{,xy} + A_{33} v^{o}_{,xx} + A_{22} v^{o}_{,yy} - B_{22} w_{,yyy} - B_{13} w_{,xxx} - 3 B_{23} w_{,yyy} = 0 \tag{49}$$

$$D_{11} w_{,xxxx} + 2 (D_{12} + 2 D_{33}) w_{,xxyy} + D_{22} w_{,yyyy} - B_{11} u^{o}_{,xxx} - B_{22} v^{o}_{,yyy} - B_{13} (v^{o}_{,xxx} + 3 u^{o}_{,xxy}) - B_{23} (3 v^{o}_{,xyy} + u^{o}_{,yyy}) = q \tag{50}$$

where u^o, v^o and w denote the components of displacement of the middle plane along the x, y and z axes, respectively, a camma indicates partial differentiation.

Consider a rectangular cross-ply composite of dimensions a, b having an even number of plies with each ply having the same thickness alternately oriented at 0^o and 90^o to the x - y axes of the plate.

$$A_{22} = A_{11},\ D_{22} = D_{11},\ B_{11} = -B_{22} = \frac{(E_L - E_T)}{8\ E_T\ N}\ h^2\ K_{11},$$

$$B_{13} = B_{23} = 0 \qquad (51)$$

Assume that the plate is simply-supported in such a manner that normal displacements are admissible, but lateral (tangential) displacements are not. Thus the following boundary conditions are appropriate:

$$\begin{aligned} w\ (0,\ y) &= w\ (a,\ y) = w\ (x,\ 0) = w\ (x,\ b) \\ &= M_x\ (0,\ y) = M_x\ (a,\ y) = M_y\ (x,\ 0) \\ &= M_y\ (x,\ b) = 0 \end{aligned} \qquad (52)$$

$$\begin{aligned} v^o\ (0,\ y) &= v^o\ (a,\ y) = u^o\ (x,\ 0) = u^o\ (x,\ b) \\ &= N_x\ (0,\ y) = N_x\ (a,\ y) = N_y\ (x,\ 0) \\ &= N_y\ (x,\ b) = 0 \end{aligned} \qquad (53)$$

A solution can be obtained by expanding the transverse load q into a double Fourier series as

$$q = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \qquad (54)$$

and solutions of the form

$$u^o = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \qquad (55)$$

$$v^o = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \qquad (56)$$

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{57}$$

The boundary conditions, equations (52), (53) are identically satisfied by the displacement field, equations (55) to (57). Substituting equations (54) to (57) into the governing equations (48) to (50), taking equation (51) into account, and solving the resulting simultaneous equations, for the Fourier coefficients, yields

$$A_{mn} = \frac{q_{mn} R^3 b^3 B_{11} n}{\pi^3 D_{mn}} (A_{33} m^4 + A_{11} m^2 n^2 R^2 + (A_{12} + A_{33}) n^4 R^4) \tag{58}$$

$$B_{mn} = - \frac{q_{mn} R^4 b^3 B_{11} n}{\pi^3 D_{mn}} ((A_{12} + A_{33}) m^4 + A_{11} m^2 n^2 R^2 + A_{33} n^4 R^4) \tag{59}$$

$$C_{mn} = \frac{q_{mn} R^4 b^4}{\pi^4 D_{mn}} ((A_{11} m^2 + A_{33} n^2 R^2) (A_{33} m^2 + A_{11} n^2 R^2) - (A_{12} + A_{33})^2 m^2 n^2 R^2) \tag{60}$$

Where R is the length to width ratio of the plate, a/b,

$$\begin{aligned} \text{and} \quad D_{mn} = & (((A_{11}\, m^2 + A_{33}\, n^2\, R^2)(A_{33}\, m^2 + A_{11}\, n^2\, R^2) \\ & - (A_{12} + A_{33})^2\, m^2\, n^2\, R^2)\ (D_{11}\ (m^4 + n^4\, R^4) \\ & + 2\ (D_{12} + 2\, D_{33})\ m^2\, n^2\, R^2) - B_{11}^2 (A_{11}\, m^2\, n^2\, R^2 \\ & * (m^4 + n^4\, R^4) + 2\ (A_{12} + A_{33})\ m^4\, n^4\, R^4 \\ & + A_{33}\ (m^8 + n^8\, R^8))) \end{aligned} \tag{61}$$

If the load q is a sinusoidal distributed load as given by

$$q\ (x,\ y) = q_o \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \tag{62}$$

and the plate is a square laminated orthotropic plate i.e. coupling matrix $[B]$ is identically zero, then for a unit load q_o i.e. $q_o = 1$ the maximum transverse deflection at the middle of the plate is given by

$$w_{max} = \frac{a^4}{2\,\pi^4\ (D_{11} + D_{12} + 2\, D_{33})} \tag{63}$$

because $m = n = 1$ in equation (62).

2.3.2 Buckling Analysis of Laminated Plates

The differential equation describing the bending of an orthotropic laminated plate subjected to an in-plane edge loading (neglecting any bending-membrane

coupling term) is [6, 7]

$$D_{11}\, w_{,xxxx} + 4\, D_{13}\, w_{,xxxy} + 2\, D_3\, w_{,xxyy} + 4\, D_{23}\, w_{,xyyy}$$

$$+ D_{22}\, w_{,yyyy} + N_x\, w_{,xx} + 2\, N_{xy}\, w_{,xy} + N_y\, w_{,yy} = 0 \tag{64}$$

For the orthotropic case with the principal axes parallel to the sides of the plate, and with $N_{xy} = N_y = 0$, equation (64) becomes

$$D_{11}\, w_{,xxxx} + 2\, D_3\, w_{,xxyy} + D_{22}\, w_{,yyyy} + N_x\, w_{,xx} = 0 \tag{65}$$

where $D_3 = D_{12} + 2\, D_{33}$.

In the case of a rectangular plate with all the four edges simply-supported, the mode shape is assumed to be

$$w = A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{66}$$

substituting equation (66) into equation (65), the critical buckling load can be found to be

$$(N_x)_{cr} = \frac{\pi^2}{b^2} \left[D_{11} \frac{m^2\, b^2}{a^2} + 2\, D_3\, n^2 + D_{22} \frac{a^2\, n^2}{m^2\, b^2} \right] \tag{67}$$

where m and n are chosen to make $(N_x)_{cr}$ a minimum. Since n appears only in the numerator, the minimum occurs at $n = 1$.

For an isotropic plate $D_{11} = D_{22} = D_3 = \dfrac{E\,h^3}{12(1-\mu^2)} = D$

and equation 67 becomes

$$(N_x)_{cr} = \frac{\pi^2 D}{b^2}\left(\frac{mb}{a} + \frac{a}{mb}\right)^2 \qquad (68)$$

and, for a square isotropic plate,

$$(N_x)_{cr} = \frac{4\,\pi^2 D}{b^2} \qquad (69)$$

In case the plate has unbalanced configuration the stiffness coupling matrix $[B]$ will be non-zero. If the coupling matrix has non-zero terms, the preceding eigen value formulation can still be used if the effect of the coupling terms can be approximated by adjusting the flexural stiffnesses. This can be accomplished by first considering the force-deformation equations (1).

These equations (1) can be solved for $\{M\}$ by solving the first equation for ε_o and then substituting its value into the second equation as follows:

$$\{N\} = [A]\{\varepsilon_o\} - [B]\{\chi\}$$

$$\{\varepsilon_o\} = [A]^{-1}\{N\} + [A]^{-1}[B]\{\chi\}$$

$$\{M\} = [B]\{\varepsilon_o\} - [D]\{\chi\}$$

$$\{M\} = [B][A]^{-1}\{N\} + [B][A]^{-1}[B]\{\chi\} - [D]\{\chi\}$$

$$\{M\} = ([D] - [B][A]^{-1}[B]\{-\chi\} + [B][A]^{-1}\{N\} \qquad (70)$$

$$\{M\} = [D^*]\{-X\} + [B][A]^{-1}\{N\} \tag{71}$$

where

$$[D^*] = [D] - [B][A]^{-1}[B] \tag{72}$$

The reduced stiffness matrix, $[D^*]$, can easily be calculated and used in place of $[D]$ in the buckling predictions.

The effect of the membrane-bending coupling terms on the buckling load has largely been neglected in the past. While it is quite clear from equation (71) that coupling will have very little effect in most cases because of the relative magnitudes of the $[A]^{-1}$ and $[B]$ matrices, it has been found, in some cases, the coupling terms cannot be ignored.

2.3.3 Vibration Analysis of Laminated Plates

The following discussion on the transverse vibration without in-plane forces are restricted to the rectangular, orthotropic laminates and coupled laminates with $D^*_{13} = D^*_{23} = 0$ for which the theory of homogeneous orthotropic plates is applicable. The effect of material damping, the energy-dissipation properties of a material, is not included in the analysis. The edges of the plate having sides of length a and b are assumed to be simply supported. For most cases in the vibration of rectangular plates, the modal lines of the normal mode are parallel

to the sides of the plates, and the normal mode is therefore defined by m and n, the number of nodal line in the x and y directions, respectively. This number does not include the nodal line at the plate edge, i.e. the simply supported edge.

The differential equation of motion for the plate element subjected to internally induced moments and externally applied force P as shown in Fig. 5, neglecting body force and rotary inertia, is

$$\frac{\partial^2 M_x}{\partial x^2} - 2\,\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} + P = \rho^* h\,\frac{\partial^2 w}{\partial t^2} \tag{73}$$

where ρ^* is the average mass density of the laminate. Since shear deformation is neglected, curvatures can be expressed in terms of deflection as

$$\chi_x = -\frac{\partial^2 w}{\partial x^2}\,,\quad \chi_y = -\frac{\partial^2 w}{\partial y^2}\,,\quad \chi_{xy} = 2\,\frac{\partial^2 w}{\partial x\,\partial y}$$

Therefore, the moments for the orthotropic laminate given in equation (1) in expanded form become,

$$M_x = -\left(D_{11}\,\frac{\partial^2 w}{\partial x^2} + D_{12}\,\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -\left(D_{12}\,\frac{\partial^2 w}{\partial x^2} + D_{22}\,\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_{xy} = 2\,D_{33}\,\frac{\partial^2 w}{\partial x\,\partial y}\,.$$

Introducing the above moment expressions in equation (73)

yields the following differential equation of motion for the orthotropic laminate in terms of deflection:

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 2 D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} + \rho^* h \frac{\partial^2 w}{\partial t^2} = P \tag{74}$$

The boundary conditions to be satisfied, for the simply supported edge whose normal direction is y, are

$$w = 0$$

$$M_y = 0 \quad \text{or} \quad D_{22} \frac{\partial^2 w}{\partial y^2} + D_{12} \frac{\partial^2 w}{\partial x^2} = 0 \tag{75}$$

For the coupled laminate, the moment expressions given in equation (2) lead to the above equation of motion and boundary conditions in which D_{ij} are replaced by the reduced stiffness D^*_{ij}.

The differential equation of motion and the boundary conditions given in equation (74) and (75) are identical to those of a homogeneous orthotropic plate except that the coefficients appearing in equation (74) and (75) are the average mass density and the flexural stiffnesses (or the reduced stiffness) of the laminate. Since the coefficients do not affect the form of the solutions, the solutions available for the homogeneous plates are applicable for the laminated plate if the average mass density and the proper stiffnesses (D_{ij} for the orthotropic laminate and D^*_{ij} for the coupled laminate)

are used. In the following discussion, unstarred D_{ij} will be used for simplicity.

The solution for the free vibration of the plate which is described by the homogeneous form of equation (74) leads to normal modes and corresponding natural frequencies. No difficulty is encountered in effecting a separation of spatial and time-dependent variable for the homogeneous form of equation (74).

Setting $w = \sum_m \sum_n W_{mn}(x, y)\, q_{mn}(t)$ it developes that q_{mn} is a harmonic function of time with circular frequency P_{mn} and that W_{mn} must satisfy the equation

$$D_{11} \frac{\partial^4 W_{mn}}{\partial x^4} + 2 D_3 \frac{\partial^4 W_{mn}}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 W_{mn}}{\partial y^4} = \rho^* h\, p_{mn}^2 W_{mn} \tag{76}$$

A further separation of W_{mn} as a product of a function of x and a function of y is possible for plates having two opposite edges simply supported.

Assuming

$$W_{mn} = \sin \frac{(m + 1)\pi x}{a} Y_{mn}$$

which satisfies the boundary conditions at edges $x = 0$ and $x = a$, and substituting this into equation (76) yields the following ordinary differential equation for Y_{mn} :

$$D_{22} \frac{d^4 Y_{mn}}{dy^4} - 2 D_3 \frac{(m+1) n^2}{a^2} \frac{d^2 Y_{mn}}{dy^2}$$

$$+ \left[\frac{(m+1)^4 \pi^4}{a^4} D_{11} - \rho^* h p^2_{mn} \right] Y_{mn} = 0 \tag{77}$$

The general solution of equation (7) is

$$Y_{mn} = C_{1mn} \cosh \frac{\phi_{mn} y}{a} + C_{2mn} \sinh \frac{\phi_{mn} y}{a}$$

$$+ C_{3mn} \cos \frac{\psi_{mn} y}{a} + C_{4mn} \sin \frac{\psi_{mn} y}{a} \tag{78}$$

where

$$\phi_{mn} = \frac{(m+1)\pi}{\sqrt{D_{22}}} \left\{ \left[D_3 - D_{11} D_{12} + \frac{\rho^* h D_{22} a^4 p^2_{mn}}{(m+1)^4 \pi^4} \right]^{1/2} + D_3 \right\}^{1/2}$$

$$\psi_{mn} = \frac{(m+1)\pi}{\sqrt{D_{22}}} \left\{ \left[D_3 - D_{11} D_{12} + \frac{\rho^* h D_{22} a^4 p^2_{mn}}{(m+1)^4 \pi^4} \right]^{1/2} - D_3 \right\}^{1/2} \tag{78a}$$

The parameters ϕ_{mn} and ψ_{mn} are related by the equation

$$\phi^2_{mn} - \psi^2_{mn} = 2 (m+1)^2 \pi^2 \frac{D_3}{D_{22}}$$

The constants C_{imn} are determined by four homogeneous boundary conditions, two at the edge $y = 0$ and two at the edge $y = b$, given in equation (75). For homogeneous boundary conditions, only the ratios of the C_{imn} are determined through the use of any three equations, while the fourth boundary condition leads to the

frequency equation. The roots of this equation are the eigen values ϕ_{mn} (or ψ_{mn}) for which the solution is valid. For each ϕ_{mn} there is corresponding eigenfunction W_{mn} which constitutes a normal mode of vibration of the plate. The complete solution is obtained by superposition of all the normal modes; i.e.,

$$w = \sum_m \sum_n \sin \frac{(m+1)\pi x}{a} Y_{mn}(y)\, q_{mn}(t) \qquad n, m = 0, 1, 2 \ldots \tag{79}$$

The circular frequency p_{mn} is a **natural** frequency of the plate and relates to the roots ϕ_{mn} of the frequency equation by equation(78 (a)) and is given by

$$p_{mn} = \frac{\left[D_{22}\,\phi_{mn}^4 - 2\,(m+1)^2 \pi^2 D_3\,\phi_{mn}^4 + (m+1)^4 \pi^4 D_{11}\right]^{1/2}}{a^2 \sqrt{\rho^* h}}$$

The frequency equation for a plate simply supported on all the four edges is given by

$$\sin \psi_{mn} R = 0,$$

the roots of which are $\psi_{mn} = \dfrac{(n+1)\pi}{R} = \dfrac{(n+1)\pi a}{b}$,

where **R** is the aspect ratio of the plate.

Therefore the natural frequencies are given by

$$p_{mn} = \frac{\pi^2}{\sqrt{\rho^* h}} \left[\frac{(m+1)^4}{a^4} D_{11} + 2\,\frac{(m+1)^2 (n+1)^2}{a^2 b^2} D_3 + \frac{(n+1)^4}{b^4} D_{22} \right]^{1/2} \tag{80}$$

and Y_{mn} by

$$Y_{mn} = \sin \frac{(n+1)\pi y}{b} \tag{81}$$

The nodal patterns of a normal mode consists of straight lines parallel to the plate boundaries of

(a) m lines given by $\sin \frac{(n+1)\pi x}{a} = 0$ in the x - direction.

(b) n lines given by $\sin \frac{(n+1)\pi y}{b} = 0$ in the y - direction.

The wave forms in the x - and y - directions for SSSS boundary conditions are sine waves and are independent of the values of the elastic moduli.

CHAPTER III

FORMULATION AND SOLUTION OF OPTIMISATION PROBLEM

When a means for predicting the behavior of any design within a particular design concept is available, limitations on the performance and other external constraints on the design can be stated, and an acceptance criterion can be established, it is possible to cast the design modification problem in the form of a mathematical programming problem, which can be stated as follows:

Find $\bar{X}$ which minimizes $F(\bar{X})$

subject to the constraints

$$g_j(\bar{X}) \leqslant 0, \quad j = 1, 2, \ldots, p \tag{82}$$

where $\bar{X}$ is called the design vector consisting of the design variables $x_1, x_2, \ldots x_q$, $F(\bar{X})$ is called the objective function and $g_j(\bar{X})$ is the j^{th} constraint function.

3.1 PROBLEM STATEMENT

3.1.1 For Laminated Beams

In this case, the design determination of the cross-section of a simply supported beam under uniformly distributed load is considered for minimum volume when the load and the elastic properties of the beam are

specified. Constraints are specified on the normal and shear strengths and on the maximum deflection.

More specifically, the following optimisation problems were considered.

Problem I:

Given:

1. The available orientation angles θ_i and the corresponding material properties E_{Li}, E_{Ti}, G_{LTi} and ν_{LTi} for i = 1, 2, ..., N and noting that ($\nu_{LTi} / E_{Li} = \nu_{TLi}/E_{Ti}$).
2. N the number of available orientation angles, i.e. half the number of plies.
3. Upper limits on the normal and shear stresses in laminate as X, Y and S respectively.
4. Load distribution on the beam.
5. Span 'l' of the beam, width of the beam h = 1.

Find : A set of t_i, i = 1, 2, ..., N, such that

1. Volume $= 2l \sum_{i=1}^{N} t_i \longrightarrow$ minimum (83)

2. $\sigma_x \big|_{max} \leqslant X$ (84)

3. $\sigma_y \big|_{mas} \leqslant Y$ (85)

4. $\tau_{xy} \big|_{max} \leqslant S$ (86)

5. $t_i \geqslant 0$ for i = 1, 2, ..., N (87)

+ If, as is often the case, the laminate is to be made from a single type of fibre composite material, only one set of material properties need be specified.

It should be noted that equation (83) states the minimum volume objective, equations (84) to (86) implement the stress constraints and equation (87) requires that the thicknesses be non-negative.

Problem II:

Given:

1. The available orientation angles θ_i and the corresponding material properties E_{Li}, E_{Ti}, G_{LTi} and ν_{LTi} for i = 1, 2, ..., N.
2. N the number of available orientation angles.
3. Upper limits on the normal and shear stresses in laminate as X, Y and S respectively.
4. Upper limit on the maximum transverse deflection of the beam.
5. Load distribution on the beam.
6. Span 'l' of the beam, width of the beam h = 1.

Find: A set of t_i, i = 1, 2, ..., N, such that

1. $$\text{Volume} = 2l \sum_{i=1}^{N} t_i \longrightarrow \text{minimum} \tag{88}$$

2. $$\sigma_x\big|_{max} \leqslant X \tag{89}$$

3. $$\sigma_y\big|_{max} \leqslant Y \tag{90}$$

4. $$\tau_{xy}\big|_{max} \leqslant S \tag{91}$$

5. $v_{max} \leq v_o$ (92)

6. $t_i \geq 0$ for $i = 1, 2, \ldots, N$ (93)

It should be noted that equation (88) states the minimum weight objective, equation (89) to (91) implement the stress constraints, equation (92) implements deflection constraint and equation (93) requires that the thickness be non-negative.

It can be noticed that the above two problems are similar to the one stated in equation (82).

3.1.2 For Laminated Plates

In this work, the ply thickness (t_i) / half the number of plies (N) are treated as design variables for minimum weight design of laminates for preassigned values of frequency, buckling load and deflection. The elastic constants were taken as preassigned values, i.e., the material to be used will be selected in advance from a set of available well characterized systems. Closed form solutions for the natural frequencies, buckling loads and deflection are available only for simply supported laminated orthotropic plates.

The optimisation procedure was used for the design of simply supported laminated plates only since the computer time required on IBM 7044 becomes very

large for solving the partial differential equations for other boundary conditions by approximate methods.

The following optimisation problems were solved in the present work:

Problem I:

Given:

1. The available orientation angles θ_i and the corresponding material properties E_{Li}, E_{Ti}, G_{LTi}, ν_{LTi} and ρ_i the weight density for i = 1, 2, ..., N.

2. N the number of available orientation angles.

3. Lower limits on the laminate fundamental frequency β_j, j = 1, 2, ... k.

4. k the number of sub-problems to be solved.

5. Length 'a' and breadth 'b' of the laminated plate.

Find : A set of t_i, i = 1, 2, ..., N, such that

1. $$\text{Weight} = 2 \sum_{i=1}^{N} \rho_i \, t_i \, ab \rightarrow \text{minimum} \qquad (94)$$

2. $$p_{mn} \geqslant \beta_j \; , \quad j = 1, 2, \ldots, k \qquad (95)$$

3. $$t_i \geqslant 0 \quad \text{for } i = 1, 2, \ldots, N \qquad (96)$$

It should be recognized that equation (94) states the minimum weight objective, equation (95) implements the frequency constraint and equation (96) requires that the thicknesses be non-negative.

Problem II :

Given :

1. The available orientation angles θ_i and the corresponding material properties E_{Li}, E_{Ti}, G_{LTi}, ν_{LTi} and ρ_i for $i = 1, 2, \ldots, N_j$.

2. N_j, $j = 1, 2, \ldots, k$, the number of available orientation angles.

3. k is the number of sub-problems to be solved.

4. Lower limit on the laminate fundamental frequency,

5. Length 'a' and breadth 'b' of the laminated plate.

Find: A set of t_i, $i = 1, 2, \ldots, N_j$ such that

1. $$\text{Weight} = 2 \sum_{i=1}^{N_j} \rho_i \, t_i \, ab \longrightarrow \text{minimum} \qquad (97)$$

2. $$p_{mn} \geq \beta \qquad (98)$$

3. $$t_i \geq 0 \quad \text{for } i = 1, 2, \ldots, N_j \qquad (99)$$

The equations (97) to (99) have the same meaning as equations (94) to (96) of problem I.

Problem III :

Given :

1. The available orientation angles θ_i and the corresponding material properties E_{Li}, E_{Ti}, G_{LTi}, ν_{LTi} and ρ_i for $i = 1, 2, \ldots, N$.

2. N the number of available orientation angles.
3. Upper limit on the weight of the plate
   W_j, $j = 1, 2, \ldots, k$
4. k is the number of sub-problems to be solved.
5. Length 'a' and breadth 'b' of the laminated plate.

Find: A set of t_i, $i = 1, 2, \ldots, N$, such that

1. Frequency = $p_{mn} \rightarrow$ maximum (100)
2. weight $\leqslant W_j$ (101)
3. $t_i \geqslant 0$ for $i = 1, 2, \ldots, N$ (102)

It should be recognised that equation (100) states the maximum frequency objective, equation (101) implements the weight constraint and equation (102) requires that the thicknesses be non-negative.

Problem IV:

Given:

1. The available orientation angles θ_i and the corresponding material properties E_{Li}, E_{Ti}, G_{LTi}, ν_{LTi} and ρ_i for $i = 1, 2, \ldots, N_j$.
2. N_j the number of available orientation angles for $j = 1, 2, \ldots, k$.
3. k is the number of sub-problems to be solved.
4. Lower limits on the laminate fundamental frequency β_j, and the critical buckling load α_j.

5. Length 'a' and breadth 'b' of the laminated plate.

Find: A set of t_i, $i = 1, 2, \ldots, N_j$

1. $\text{weight} = 2 \sum_{i=1}^{N_i} \rho_i t_i ab \longrightarrow \text{minimum}$ (103)

2. $p_{mn} \geqslant \beta_j$ (104)

3. $(N_x)_{cr} \geqslant \alpha_j$ (105)

4. $t_i \geqslant 0$ for $i = 1, 2, \ldots, N_j$ (106)

It should be recognized that equation (103) states the minimum weight objective, equation (104) implements the frequency constraint and equation (105) implements the buckling load constraint and equation (106) requires that the thicknesses be non-negative.

Problem V:

Given

1. The available orientation angle θ_i and the corresponding material properties E_{Li}, E_{Ti}, G_{LTi}, ν_{LTi} and ρ_i for $i = 1, 2, \ldots, N_j$.
2. N_j the number of available orientation angles for $j = 1, 2, \ldots, k$.
3. k is the number of sub-problems to be solved.

4. Lower limits on the laminate fundamental frequency, β, and the critical buckling load, α.

5. Upper limit on the transverse deflection of the plate, w_{max}.

6. Length 'a' and breadth 'b' of the laminated plate.

Find: A set of t_i, $i = 1, 2, \ldots, N_j$, such that

1. $$\text{Weight} = 2 \sum_{i=1}^{N_j} \rho_i \, t_i \, ab \qquad \text{minimum} \tag{107}$$

2. $$p_{mn} \geqslant \beta \tag{108}$$

3. $$(N_x)_{cr} \geqslant \alpha \tag{109}$$

4. $$w \leqslant w_{max} \tag{110}$$

5. $$t_i \leqslant 0 \text{ for } i = 1, 2, \ldots, N_j \tag{111}$$

Here equation (107) states the minimum weight objective, equation (108) implements the frequency constraint, equation (109) implements critical buckling load constraint, equation (110) implements the deflection constraint, and equation (111) requires that the thicknesses be non-negative.

It can be noticed that all these five problems are similar to the one stated in equation (82).

3.2 SOLUTION TECHNIQUE

An examination of the frequency, buckling load and the deflection equations reveals that the quantities are expressed in terms of the material stiffnesses A_{ij}, coupling matrix elements B_{ij}, and flexural rigidities D_{ij}, etc., which in turn are functions of the thickness, orientation and properties of each ply. Since B_{ij} and D_{ij} are not linear functions of the ply thicknesses, these quantities will be non-linear functions of the design vector i.e., ply thicknesses. The non-linear optimization problems are solved using the interior penalty function method in this work.

In this approach, we augment the objective function with a penalty term which is small at points away from the constraints in the feasible region, but which 'blows up' as the constraints are approached. It can be expressed as

$$\phi\,(\,\overline{X},\ r) \;=\; F\,(\,\overline{X}\,) - r \sum_{j=1}^{p} 1/g_j\,(\,\overline{X}\,) \qquad (112)$$

where F is to be minimized over all $\overline{X}$ satisfying $g_j\,(\,\overline{X}\,) \leqslant 0$, $j = 1, 2, \ldots, p$. Note that if r is positive the effect is to add a positive penalty to $F\,(\,\overline{X}\,)$. This is because at an interior point all the terms in the sum are negative. As a boundary is approached, some g_j will approach zero and the penalty will "explode". It has been

proved that the constrained minimum of F can be obtained by finding the unconstrained minimum of $\emptyset$ for decreasing sequence of r - values.

3.2.1 Algorithm

The solution procedure can be described by the following steps:

1. Given a starting point $\bar{X}_o$ satisfying all $g_j(\bar{X}) \leqslant 0$ and an initial value of r, minimize $\emptyset$ to obtain $\bar{X}_M$.
2. Check for convergence of $\bar{X}_M$ to the optimum.
3. If the convergence criterion is not satisfied replace r by $r \leftarrow rc$ where $c < 1$.
4. Take $\bar{X}_M$ as the new starting point and repeat from step 1.

There are a number of points to consider in applying the method:

a) The starting design $\bar{X}_o$ required by step 1 is usually available in engineering problems, but sometimes finding such a point becomes difficult.

b) A proper initial value for r must be selected.

c) The possibilities for the convergence criteria of step 2 are numerous and there are choices to be made.

d) Because of the sequential nature of the process, it is possible to improve the starting point for the third and subsequent minimizations.

3.2.2 Starting Point

In many engineering situations, particularly in the structural and mechanical design areas, it is easy to find a point satisfying $g_j (\bar{X}) \leqslant 0$ at the expense of large values of F. For example, in structural design, if cost or weight of the structure is ignored, it is usually easy to propose many designs which fulfill the basic requirements of strength and rigidity for the particular application. In other design situations, however, the acceptable designs may not be at all obvious. In these situations the initial acceptable design required by the interior penalty function method can be obtained as follows.

Suppose an engineering assessment of the situation has produced the design $\bar{X}_o$ which satisfies $g_j (\bar{X}_o) < 0$, $j = 1, 2, \ldots, s$, but which has $g_j (\bar{X}_o) > 0$, $j = s + 1, s + 2, \ldots, p$. Take the k for which $g_k (\bar{X}_o)$ is a maximum out of $g_j (\bar{X}_o)$ for $j = s + 1, s + 2, \ldots, p$, and temporarily define it as the objective function for the following problem :

Find $\bar{X}$ such that

$$g_k(\bar{X}) \rightarrow \text{minimum}$$

$$g_j(\bar{X}) \leqslant 0 \qquad j = 1, 2, \ldots, s$$

$$g_j(\bar{X}) - g_j(\bar{X}_o) \leqslant 0, \quad j = s + 1, s + 2, \ldots$$
$$k - 1, k + 1, \ldots, p$$

Whenever, during the process of solving this problem by the penalty function method, the value of $g_k(\bar{X})$ drop below zero, the procedure is halted. The point so obtained then satisfies at least one more constraint than did the original $\bar{X}_o$. The procedure can be repeated untill all the constraints have been satisfied and an $\bar{X}_o$ is obtained for which $g_j(\bar{X}_o) < 0$, $g = 1, 2, \ldots, p$.

3.2.3 Initial Value for r

The matter of selecting an initial value for the penalty parameter r has been discussed in the literature, but while some theory is available, the task is still an art. If r is large, the function is easy to minimize, but the minimum may lie far from the desired solution to the original constrained minimization problem. On the other hand, if r is small, the function will be hard to minimize. A general rule is that if $\bar{X}_o$ is a conservative design, pick r_o so that $-r_o \sum 1/g_j(\bar{X}_o)$ approximately equals $\frac{1}{2} F(\bar{X}_o)$. In practice, this approach usually yields a reasonable initial value for r.

3.2.4 Convergence Criteria

As the $\emptyset$ - function is minimized for various decreasing values of r, the sequence of minima $\bar{X}_M(r_i)$, i = 1, 2, ..., should converge to the solution of the constrained minimization problem. A means is needed to ascertain this convergence without an unnecessarily large number of minimizations. One simple criterion is to compute the relative difference

$$\delta = \frac{F_{min}(r_{i-1}) - F_{min}(r_i)}{F_{min}(r_i)} \tag{113}$$

and stop when this value drops below a certain fraction.

Another convergence test can be stated as

$$|\bar{\Delta}| \leqslant \varepsilon$$

where $$\bar{\Delta} = \bar{X}_M(r_{i-1}) - \bar{X}_M(r_i) \tag{114}$$

3.2.5 Unconstrained Minimization

For the unconstrained minimization of the penalty function, the Variable Matric method (also known as the Davidon - Fletcher - Powel method) has been used. This method is a first order method which replaces the local hessian J_q^{-1} by an approximate metric H_q. The method of computing this metric completely eleminates the need for evaluating the second derivatives and performing matrix inversions, and yet the sequence of iterations converges quadratically to the minimum point $\bar{X}_M$.

Furthermore, it can be proved that the matric H_q, which is improved at each iteration, converges to J_M^{-1} .

The iterative process can be stated as follows:

1. Start with an initial $\bar{X}_o$ and an initial positive definite symmetric matrix H_o (for example, the identity matrix), and set $\bar{S}_o = - H_o \nabla F_o$.

2. Compute $\bar{X}_{q+1} = \bar{X}_q + \alpha^*_q \bar{S}_q$, where α^*_q minimizes $F(\bar{X}_q + \alpha \bar{S}_q)$, and check for convergence.

3. Compute $H_{q+1} = H_q + M_q + N_q$, where, defining

$$\bar{Y}_q = \bar{G}_{q+1} - \bar{G}_q \equiv \nabla F(\bar{X}_{q+1}) - \nabla F(\bar{X}_q)$$

$$M_q = \alpha^*_q \frac{\bar{S}_q \bar{S}_q^T}{\bar{S}_q^T \bar{Y}_q}$$

and $$N_q = - \frac{(H_q \bar{Y}_q)(H_q \bar{Y}_q)^T}{\bar{Y}_q^T H_q \bar{Y}_q}$$

4. Compute $\bar{S}_{q+1} = - H_{q+1} \cdot \bar{G}_{q+1}$ and repeat from step 2.

3.3 NUMERICAL RESULTS

The numerical results of optimizations are presented in this section. The numerical data assumed for each ply in all the problems was

$$E_L = 31.0 \times 10^6 \text{ psi}; \quad E_T = 3.4 \times 10^6 \text{ psi};$$
$$G_{LT} = 0.75 \times 10^6 \text{ psi}; \quad \nu_{LT} = 0.28 \text{ and}$$
$$\rho = .0001856 \text{ lb/in}^3.$$

The orientation angles assumed were 0° and 90° alternatively with symmetry about the middle plane.

3.3.1 Laminated Beams

The common data for the two problems is

$l = 10.0$ in., $X = 48{,}000$ psi, $Y = 30{,}000$ psi, $S = 8{,}000$ psi.

Problem I:

The number of design variables considered is $N = 2$ only and the strength constraints were cubic functions of t_1 and t_2 so the problem was solved graphically.

Objective : minimization of volume (in.^{3})

Design variables : thicknesses of plies (in.)

The values of t_2 for preassigned values of t_1 satisfying the various strength constraints are listed in table 3.1, and the equations are shown plotted in figure 6.

TABLE 3.1 Values of t_2 for different t_1

t_1 (inches)	Value of t_2 from constraint on (inch.) Normal stress σ_x	Normal stress σ_y	Shear stress τ_{xy}
0.0	0.09842	0.12450	0.21307
0.005	0.01890	0.02992	0.08039
0.010	0.00804	0.01438	0.04422
0.015	0.00227	0.00719	0.02890
0.020	- 0.00278	0.00183	0.01979
0.025	(-)ve values	- 0.00320	0.01310
0.030	"	(-)ve values	0.00749
0.035	"	"	0.00235
0.040	"	"	- 0.00265

Lines representing constant volume are also plotted in figure 6, and the optimum thicknesses are found to be

$t_1 = 0.0375$ in., $t_2 = 0.0$ in and

volume $= 0.75$ in^3.

It can be seen that the shear stress constraint is active at the optimum point. This can be used as the optimality criterion for this class of beams.

Problem II

Though the number of design variables is $N = 2$ in this case also, it is not possible to express the deflection constraint in explicit form in terms of the thicknesses t_1 and t_2, as it involved the inverse of $[A]$. Hence the problem is solved on IBM 7044 computer using the penalty function technique. In all, five values of r were considered for the minimization of ϕ - function with 10 iterations for each r (in the Davidon - Fletcher - Powell method). Three refits were permitted during each cubic interpolation stage.

Load acting on the beam $Q = 1.24$ lb/in^2

Upper limit on deflection, $v_{max} = 0.1$ in.

Objective : Minimization of volume (in.^{3})

Design variables : Thickness of plies (inch.)

At starting design point :

$\bar{X}_o = 0.375, \quad 0.01$

$r_1 = 0.07$

$\phi(\bar{X}_o, r_1) = 15.38384$

$F(\bar{X}_o) = 7.7$

$v(\bar{X}_o) = 0.0755444$ inch.

$\sigma_x(\bar{X}_o) = 441.87559$ lb/in.^{2}

$\sigma_y(\bar{X}_o) = 123.72517$ lb/in.^{2}

$$\tau_{xy} (\bar{X}_o) = 0.2409775 \times 10^{-6} \text{ lb/in.}^2$$

At final design point :

$$\bar{X}_{opt} = 0.321125, \ 0.0469442$$

$$r_5 = 0.7 \times 10^{-6}$$

$$\phi (\bar{X}_{opt}, r_5) = 7.361556$$

$$F (\bar{X}_{opt}) = 7.361383$$

$$v (\bar{X}_{opt}) = 0.09954 5503 \text{ inch}$$

$$\sigma_x (\bar{X}_{opt}) = 484.32 \text{ lb/in.}^2$$

$$\sigma_y (\bar{X}_{opt}) = 135.6 \text{ lb/in}^2$$

$$\tau_{xy} (\bar{X}_{opt}) = 0.2409775 \times 10^{-6} \text{ lb/in.}^2$$

Throughout the optimization process, the deflection constraint has been active.

A comparison of the two beam problems indicates that the deflection is a critical parameter in the design of beams as one-tenth of the load resulted in about nine times increase in volume of the beam when deflection is also considered as a constraint.

3.3.2 Laminated Plates

The results for all the problems were obtained on IBM 7044 computer. The numerical results of optimization are given in this section. In all the problems, four values of r were considered for the minimisation of ϕ - function with 10 iterations for each r (in the Davidon - Fletcher - Powell method). Three refits were permitted during each cubic interpolation stage.

Problem I

Total number of plies = 8

Number of design variables, N = 4

Objective : minimization of weight (lbs.)

Design variables: thicknesses of plies (inch)

(a) β_1 = 100.0 cycles/sec.

At starting design point :

$\bar{X}_o$ = 0.012, 0.012, 0.012, 0.012

$r_1 = 1 \times 10^{-6}$

$\phi(\bar{X}_o, r_1) = 0.2657411 \times 10^{-2}$

$F(\bar{X}_o) = 0.1990018 \times 10^{-2}$

$p_{11}(\bar{X}_o) = 237.61335$ cycles/sec.

At final design point :

$\overline{X}_{opt}$ = 0.0050566, 0.00508049, 0.00505441, 0.00507321

r_4 = 1×10^{-9}

$\phi (\overline{X}_{opt}, r_4)$ = 0.8420442×10^{-3}

$F (\overline{X}_{opt})$ = 0.8401486×10^{-3}

$p_{11} (\overline{X}_{opt})$ = 100.315948

(b) β_2 = 150.0 cycles/sec.

At starting design point :

$\overline{X}_o$ = 0.012, 0.012, 0.012, 0.012

r_1 = 1×10^{-6}

$\phi (\overline{X}_o, r_1)$ = 0.2658396×10^{-2}

$F (\overline{X}_o)$ = 0.1990018×10^{-2}

$p_{11} (\overline{X}_o)$ = 237.61335 cycles/sec.

At final design point :

$\overline{X}_{opt}$ = 0.00733184, 0.00772892, 0.00765082, 0.00763795

r_4 = 1×10^{-9}

$\phi (\overline{X}_{opt}, r_4)$ = 0.1259935×10^{-2}

$F (\overline{X}_{opt})$ = 0.1258252×10^{-2}

$p_{11} (\overline{X}_{opt})$ = 150.238555 cycles/sec.

(c) β_3 = 200.0 cycles/sec.

At starting design point :

$\overline{X}_o$ = 0.012, 0.012, 0.012, 0.012

r_1 = 1×10^{-6}

$\phi (\overline{X}_o, r_1)$ = 0.2662002×10^{-2}

$F (\overline{X}_o)$ = 0.1990018×10^{-2}

$p_{11} (\overline{X}_o)$ = 237.61335 cycles/sec.

At final design point

$\overline{X}_{opt}$ = 0.0100441, 0.0103504, 0.0100447, 0.0100446

r_4 = 1×10^{-9}

$\phi (\overline{X}_{opt}, r_4)$ = 0.1679688×10^{-2}

$F (\overline{X}_{opt})$ = 0.1678405×10^{-2}

$p_{11} (\overline{X}_{opt})$ = 200.405934 cycles/sec.

(d) β_4 = 250.0 cycles/sec.

At starting design point :

$\overline{X}_o$ = 0.015, 0.015, 0.015, 0.015

r_1 = 1×10^{-6}

$\phi (\overline{X}_o, r_1)$ = 0.3026173×10^{-2}

$F (\overline{X}_o)$ = 0.2487522×10^{-2}

$p_{11} (\overline{X}_o)$ = 297.01669 cycles/sec.

The relation between minimum weight and β_j is shown plotted in figure 7(a). It can be seen that the minimum weight varies linearly with the lower bound value of the frequency. The computer time taken in each case is about eight minutes. It was observed that frequency constraint was active in all the problems.

Problem II

β = 200.0 cycles/sec.

Objective : minimization of weight (lbs.)

Design variables : thicknesses of plies (inch)

(a) Total number of plies = 4

Number of design variables, $N_1 = 2$

At starting design point :

$\overline{X}_o$ = 0.024, 0.024

$r_1 = 1 \times 10^{-6}$

$\phi(\overline{X}_o, r_1) = 0.2162002 \times 10^{-2}$

$F(\overline{X}_o) = 0.1980019 \times 10^{-2}$

$p_{11}(\overline{X}_o)$ = 237.61335 cycles/sec.

At final design point :

$\overline{X}_{opt}$ = 0.0203055, 0.0203077

$r_4 = 1 \times 10^{-9}$

$\phi(\overline{X}_{opt}, r_4) = 0.1684155 \times 10^{-2}$

$F(\overline{X}_{opt}) = 0.1683767 \times 10^{-2}$

$p_{11}(\overline{X}_o)$ = 201.046202 cycles/sec.

(b) Total number of plies = 8

Number of design variables, $N_2 = 4$

At starting design point:

$\overline{X}_o = 0.012, 0.012, 0.012, 0.012$

$r_1 = 1 \times 10^{-6}$

$\phi (\overline{X}_o, r_1) = 0.2662002 \times 10^{-2}$

$F (\overline{X}_o) = 0.1990018 \times 10^{-2}$

$p_{11} (\overline{X}_o) = 237.61335$ cycles/sec.

At final design point :

$\overline{X}_{opt} = 0.0100441, 0.0103504, 0.0100447, 0.0100446$

$r_4 = 1 \times 10^{-9}$

$\phi (\overline{X}_{opt}, r_4) = 0.1679688 \times 10^{-2}$

$F (\overline{X}_{opt}) = 0.1678405 \times 10^{-2}$

$p_{11} (\overline{X}_{opt}) = 200.405934$ cycles/sec.

(c) Total number of plies = 12

Number of design variables, $N_3 = 6$

At starting design point :

$\overline{X}_o = 0.008, 0.008, 0.008, 0.008, 0.008, 0.008$

$r_1 = 1 \times 10^{-6}$

$\phi (\overline{X}_o, r_1) = 0.3495335 \times 10^{-2}$

$F (\overline{X}_o) = 0.1990018 \times 10^{-2}$

$p_{11} (\overline{X}_o) = 237.61335$ cycles/sec.

At final design point :

$\overline{X}_{opt}$ = 0.00697087, 0.00658763, 0.00676488, 0.00649378, 0.00694369, 0.00667180

r_4 = 1×10^{-9}

$\phi(\overline{X}_{opt}, r_4) = 0.1679376 \times 10^{-2}$

$F(\overline{X}_{opt}) = 0.1676285 \times 10^{-2}$

$p_{11}(\overline{X}_{opt}) = 200.1527876$ cycles/sec.

(d) Total number of plies = 16

Number of design variables, N_4 = 8

At starting design point :

$\overline{X}_o$ = 0.006, 0.006, 0.006, 0.006, 0.006, 0.006, 0.006, 0.006

$r_1 = 1 \times 10^{-6}$

$\phi(\overline{X}_o, r_1) = 0.4662002 \times 10^{-2}$

$F(\overline{X}_o) = 0.1990018 \times 10^{-2}$

$p_{11}(\overline{X}_o) = 237.61335$ cycles/sec

At final design point :

$\overline{X}_{opt}$ = 0.00507456, 0.00507839, 0.00506896, 0.00506894, 0.00506783, 0.00507637, 0.00506783, 0.00506783

$r_4 = 1 \times 10^{-9}$

$\phi(\overline{X}_{opt}, r_4) = 0.1685403 \times 10^{-2}$

$F(\overline{X}_{opt}) = 0.1682009 \times 10^{-2}$

$p_{11}(\overline{X}_{opt}) = 200.83628$ cycles/sec.

(e) Total number of plies = 24

Number of desing variables, N_5 = 12

At starting design point :

$$\bar{X}_o = 5.05 \times 10^{-3},\ 5.05 \times 10^{-3},\ 5.05 \times 10^{-3},\ 5.05 \times 10^{-3},\ 5.05 \times 10^{-3},\ 5.05 \times 10^{-3},\ 5.05 \times 10^{-3},\ 5.05 \times 10^{-3},\ 5.05 \times 10^{-3},\ 5.05 \times 10^{-3},\ 5.05 \times 10^{-3},\ 5.05 \times 10^{-3}.$$

$$r_1 = 1 \times 10^{-6}$$

$$\phi(\bar{X}_o, r_1) = 0.7266872 \times 10^{-2}$$

$$F(\bar{X}_o) = 0.2512397 \times 10^{-2}$$

$$p_{11}(\bar{X}_o) = 299.9868 \text{ cycles/sec.}$$

At final design point :

$$\bar{X}_{opt} = 3.37050 \times 10^{-3},\ 3.37050 \times 10^{-3},\ 3.37060 \times 10^{-3},\ 3.37010 \times 10^{-3},\ 3.37010 \times 10^{-3},\ 3.37010 \times 10^{-3},\ 3.37050 \times 10^{-3},\ 3.37000 \times 10^{-3},\ 3.37040 \times 10^{-3},\ 3.37010 \times 10^{-3},\ 3.37010 \times 10^{-3},\ 3.37000 \times 10^{-3}.$$

$$r_4 = 1 \times 10^{-9}$$

$$\phi(\bar{X}_{opt}, r_4) = 0.1684815 \times 10^{-2}$$

$$F(\bar{X}_{opt}) = 0.1676713 \times 10^{-2}$$

$$p_{11}(\bar{X}_{opt}) = 200.203916 \text{ cycles/sec.}$$

The relation between minimum weight and the number of plies is shown plotted in figure 7(b). It can be observed

that the change in number of plies has a negligible effect on the minimum weight for a preassigned lower bound on the natural frequency. The computer time used for the above problems varied from approximately seven minutes to thirty five minutes depending upon the number of design variables. The frequency constraint was active in all the problems.

Problem III

Total number of plies = 8

Number of design variables, N = 4

Objective : maximization of frequency (cycles/sec.)

Design variables : thicknesses of plies (in.)

(a) $W_1 = 0.8 \times 10^{-3}$ lb.

At starting design point :

$\overline{X}_o = 4.0 \times 10^{-3},\ 4.0 \times 10^{-3},\ 4.0 \times 10^{-3},\ 4.0 \times 10^{-3}$

$r_1 = 0.3 \times 10^{-1}$

$\phi\ (\overline{X}_o,\ r_1\ ) = -29.66984$

$F\ (\overline{X}_o\ ) = -59.84564$

$W\ (\overline{X}_o\ ) = 0.6633392 \times 10^{-3}$ lb.

At final design point :

$\overline{X}_{opt} = 4.79?60 \times 10^{-3},\ 4.78922 \times 10^{-3},\ 4.79181 \times 10^{-3},$
4.80043×10^{-3}

$r_4 = 0.3 \times 10^{-4}$

$\emptyset(\bar{X}_{opt}, r_4) = -71.70992$

$F(\bar{X}_{opt}) = -71.73993$

$W(\bar{X}_{opt}) = 0.79518006 \times 10^{-3}$ lb.

(b) $W_2 = 0.125 \times 10^{-2}$ lb.

At starting design point :

$\bar{X}_o = 5.0 \times 10^{-3},\ 5.0 \times 10^{-3},\ 5.0 \times 10^{-3},\ 5.0 \times 10^{-3}$

$r_1 = 0.5 \times 10^{-1}$

$\emptyset(\bar{X}_o, r_1) = -34.65830$

$F(\bar{X}_o) = -74.80682$

$W(\bar{X}_o) = 0.08291725 \times 10^{-2}$ lb.

At final design point :

$\bar{X}_{opt} = 7.47819 \times 10^{-3},\ 7.49888 \times 10^{-3},\ 7.48639 \times 10^{-3},$
7.47108×10^{-3}

$r_4 = 0.5 \times 10^{-4}$

$\emptyset(\bar{X}_{opt}, r_4) = -111.9317$

$F(\bar{X}_{opt}) = -111.9654$

$W(\bar{X}_{opt}) = 0.12410482 \times 10^{-2}$ lb.

(c) $W_3 = 0.17 \times 10^{-2}$ in.

At starting design point :

$\bar{X}_o = 7.0 \times 10^{-3},\ 7.0 \times 10^{-3},\ 7.0 \times 10^{-3},\ 7.0 \times 10^{-3}$

$r_1 = 0.2 \times 10^{-1}$

$\emptyset(\bar{X}_o, r_1) = -93.23791$

$F(\bar{X}_o) = -104.7295$

$W(\bar{X}_o) = 0.11608433 \times 10^{-2}$ lb.

At final design point :

$\bar{X}_{opt} = 1.02802 \times 10^{-2},\ 1.01796 \times 10^{-2},\ 9.95183 \times 10^{-3},$
1.03381×10^{-2}

$r_4 = 0.2 \times 10^{-4}$

$\phi(\bar{X}_{opt}, r_4) = -152.4068$

$F(\bar{X}_{opt}) = -152.4179$

$W(\bar{X}_{opt}) = 0.16994308 \times 10^{-2}$ lb.

(d) $W_4 = 0.2 \times 10^{-2}$ lb.

At starting design point :

$\bar{X}_o = 9.0 \times 10^{-3},\ 9.0 \times 10^{-3},\ 9.0 \times 10^{-3},\ 9.0 \times 10^{-3}$

$r_1 = 0.15$

$\phi(\bar{X}_o, r_1) = -67.39446$

$F(\bar{X}_o) = -134.6523$

$W(\bar{X}_o) = 0.1493514 \times 10^{-2}$ lb.

At final design point :

$\bar{X}_{opt} = 1.18643 \times 10^{-2},\ 1.20599 \times 10^{-2},\ 1.20009 \times 10^{-2},$
1.18610×10^{-2}

$r_4 = 0.15 \times 10^{-3}$

$\phi\ (\bar{X}_{opt}, r_4) = -178.6704$

$F\ (\bar{X}_{opt}) = -178.7365$

$W\ (\bar{X}_{opt}) = 0.19811522 4 \times 10^{-2}$ lb.

(e) $W_5 = 0.25 \times 10^{-2}$ lb.

At starting design point :

$\bar{X}_o = 1.0 \times 10^{-2},\ 1.0 \times 10^{-2},\ 1.0 \times 10^{-2},\ 1.0 \times 10^{-2}$

$r_1 = 0.2$

$\phi\ (\bar{X}_o, r_1) = -62.01957$

$F\ (\bar{X}_o) = -149.6136$

$W\ (\bar{X}_o) = 0.16583475 \times 10^{-2}$ lb.

At final design point :

$\bar{X}_o = 1.49244 \times 10^{-2},\ 1.49534 \times 10^{-2},\ 1.48548 \times 10^{-2},$
1.48712×10^{-2}

$r_4 = 0.2 \times 10^{-3}$

$\phi\ (\bar{X}_{opt}, r_4) = -222.8677$

$F\ (\bar{X}_{opt}) = -222.9387$

$W\ (\bar{X}_{opt}) = 0.2471096 \times 10^{-2}$ lb.

The relation between maximum frequency and the design weight is shown plotted in figure 7(c). It can be observed that the maximum frequency varies linearly with the preassigned design weight as seen in problem I. The computer time used for each problem is approximately eight minutes.

Problem IV

β = 200.0 cycles/sec.

α = 1500.00 lb/in.

Objective : minimization of weight (lbs.)

Design variables : thicknesses of plies (inch)

(a) Total number of plies = 4

Number of design variables, N_1 = 2

At starting design point :

$\bar{X}_o = 5.05 \times 10^{-2},\ 5.05 \times 10^{-2}$

$r_1 = 0.5 \times 10^{-4}$

$\phi(\bar{X}_o, r_1) = 0.6308014 \times 10^{-2}$

$F(\bar{X}_o) = 0.4187329 \times 10^{-2}$

$p_{11}(\bar{X}_o)$ = 337.77443 cycles/sec.

$(N_x)_{cr}\Big|_{\bar{X}_o}$ = 2390.3541 lb/in.

At final design point :

$\bar{X}_{opt} = 4.34760 \times 10^{-2},\ 4.34760 \times 10^{-2}$

$r_4 = 0.5 \times 10^{-7}$

$\phi(\bar{X}_{opt}, r_4) = 0.3610272 \times 10^{-2}$

$F(\bar{X}_{opt}) = 0.3604920 \times 10^{-2}$

$p_{11}(\bar{X}_{opt})$ = 325.2304 cycles/sec.

$(N_x)_{cr}\Big|_{\bar{X}_{opt}}$ = 1525.2390 lb/in.

(b) Total number of plies = 8

Number of design variables, N_2 = 4

At starting design point :

$\bar{X}_o$ = 2.525×10^{-2}, 2.525×10^{-2}, 2.525×10^{-2}, 2.525×10^{-2}

r_1 = 0.5×10^{-4}

$\phi (\bar{X}_o, r_1)$ = 0.1224861×10^{-1}

$F (\bar{X}_o)$ = 0.4187329×10^{-2}

$p_{11} (\bar{X}_o)$ = 377.77443 cycles/sec.

$(N_x)_{cr} \Big|_{\bar{X}_o}$ = 2390.3541 lb/in.

At final design point :

$\bar{X}_{opt.}$ = 2.1728×10^{-2}, 2.17378×10^{-2}, 2.17471×10^{-2}, 2.17371×10^{-2}

r_4 = 0.5×10^{-7}

$\phi (\bar{X}_{opt}, r_4)$ = 0.3617100×10^{-2}

$F (\bar{X}_{opt})$ = 0.3604835

$p_{11} (\bar{X}_{opt})$ = 325.2226 cycles/sec.

$(N_x)_{cr} \Big|_{\bar{X}_{opt}}$ = 1525.1304 lb/in.

(c) Total Number of plies = 12

Number of design variables, N_3 = 6

At starting design point :

$$\bar{X}_o = 1.7 \times 10^{-2},\ 1.7 \times 10^{-2},\ 1.7 \times 10^{-2},\ 1.7 \times 10^{-2},\ 1.7 \times 10^{-2},\ 1.7 \times 10^{-2}$$

$$r_1 = 0.5 \times 10^{-5}$$

$$\phi(\bar{X}_o, r_1) = 0.6006799 \times 10^{-2}$$

$$F(\bar{X}_o) = 0.4228788 \times 10^{-2}$$

$$p_{11}(\bar{X}_o) = 381.51478$$

$$(N_x)_{cr}\Big|_{\bar{X}_o} = 2462.0600$$

At final design point :

$$\bar{X}_{opt} = 1.44355 \times 10^{-2},\ 1.44448 \times 10^{-2},\ 1.44366 \times 10^{-2},\ 1.44395 \times 10^{-2},\ 1.44417 \times 10^{-2},\ 1.44396 \times 10^{-2}$$

$$r_4 = 0.5 \times 10^{-8}$$

$$\phi(\bar{X}_{opt}, r_4) = 0.3594827 \times 10^{-2}$$

$$F(\bar{X}_{opt}) = 0.3591885 \times 10^{-2}$$

$$p_{11}(\bar{X}_{opt}) = 324.0544$$

$$(N_x)_{cr}\Big|_{\bar{X}_{opt}} = 1508.75259$$

(d) Total Number of plies $= 16$

Number of design variables, $N_4 = 8$

At starting design point :

$\overline{X}_o = 1.3 \times 10^{-2}, 1.3 \times 10^{-2}, 1.3 \times 10^{-2}, 1.3 \times 10^{-2},$
$1.3 \times 10^{-2}, 1.3 \times 10^{-2}, 1.3 \times 10^{-2}, 1.3 \times 10^{-2}$

$r_1 = 0.5 \times 10^{-4}$

$\phi(\overline{X}_o, r_1) = 0.3520143 \times 10^{-1}$

$F(\overline{X}_o) = 0.4311705 \times 10^{-2}$

$p_{11}(\overline{X}_o) = 388.99545$ cycles/sec.

$(N_x)_{cr}\Big|_{\overline{X}_o} = 2609.7453$ lb/in.

At final design point :

$\overline{X}_{opt} = 1.0827 \times 10^{-2}, 1.08271 \times 10^{-2}, 1.08309 \times 10^{-2},$
$1.08315 \times 10^{-2}, 1.08398 \times 10^{-2}, 1.08397 \times 10^{-2},$
$1.08358 \times 10^{-2}, 1.08358 \times 10^{-2}$

$r_4 = 0.5 \times 10^{-7}$

$\phi(\overline{X}_{opt}, r_4) = 0.3637399 \times 10^{-2}$

$F(\overline{X}_{opt}) = 0.3593131 \times 10^{-2}$

$p_{11}(\overline{X}_{opt}) = 324.1674$ cycles/sec.

$(N_x)_{cr}\Big|_{\overline{X}_{opt}} = 1510.3240$ lb/in.

(e) Total number of plies = 20

Number of design variables, N_5 = 10

At starting design point :

$\overline{X}_o$ = 1.01×10^{-2}, 1.01×10^{-2}, 1.01×10^{-2}, 1.01×10^{-2}, 1.01×10^{-2}, 1.01×10^{-2}, 1.01×10^{-2}, 1.01×10^{-2}, 1.01×10^{-2}, 1.01×10^{-2}

r_1 = 0.5×10^{-4}

$\phi(\overline{X}_o, r_1) = 0.5383276 \times 10^{-1}$

$F(\overline{X}_o) = 0.4187329 \times 10^{-2}$

$p_{11}(\overline{X}_o)$ = 377.77443 cycles/sec.

$(N_x)_{cr}\Big|_{\overline{X}_o}$ = 2390.354 lb/in

At final design point :

$\overline{X}_{opt}$ = 8.70907×10^{-3}, 8.70907×10^{-3}, 8.70854×10^{-3}, 8.71445×10^{-3}, 8.70679×10^{-3}, 8.72347×10^{-3}, 8.71662×10^{-3}, 8.71035×10^{-3}, 8.72122×10^{-3}, 8.72304×10^{-3}

r_4 = 0.5×10^{-4}

$\phi(\overline{X}_{opt}, r_4) = 0.3672401 \times 10^{-2}$

$F(\overline{X}_{opt}) = 0.3612819 \times 10^{-2}$

$p_{11}(\overline{X}_{opt})$ = 325.9430 cycles/sec.

$(N_x)_{cr}\Big|_{\overline{X}_{opt}}$ = 1535.2872 lb/in.

The relationship between the minimum weight and the number of plies is shown plotted in figure 7(d). It can be observed that the minimum weight varies with the change in number of plies for preassigned values of lower bounds on natural frequency. and critical buckling load. The optimum number of plies seems to be 14 (twice N). The computer time used for each problem varied from about seven minutes to thirtyfive minutes depending upon the number of design variables. The critical buckling load constraint was active at optimum point in all the cases.

Problem V

β = 200.0 cps.

α = 1,500 lb/in.

$w_{max} = 0.4 \times 10^{-2}$ in

Objective : minimization of weight (lb.)

Design variables : thickness of plies (in.)

(a) Total number of plies = 8

Number of design variables, N_1 = 4

At starting design point :

$\bar{X}_o = 2.5 \times 10^{-2},\ 2.5 \times 10^{-2},\ 2.5 \times 10^{-2},\ 2.5 \times 10^{-2}$

$r_1 = 0.3 \times 10^{-5}$

$\phi(\bar{X}_o, r_1) = 0.4737912 \times 10^{-2}$

$F(\bar{X}_o) = 0.4145870 \times 10^{-2}$

$p_{11}\ (\bar{X}_o\ ) = 374.03409$ cycles/sec.

$(N_x)_{cr}\Big|_{\bar{X}_o} = 2320.0541$ lb./in.

$w_{max}\ (\bar{X}_o) = 0.38836152 \times 10^{-2}$ in.

At final design point :

$\bar{X}_{opt} = 2.90605 \times 10^{-2},\ 1.99260 \times 10^{-2},\ 2.43860 \times 10^{-2},$
2.32236×10^{-2}

$r_4 = 0.3 \times 10^{-8}$

$\phi\ (\bar{X}_{opt},\ r_4) = 0.4005675 \times 10^{-2}$

$F\ (\bar{X}_{opt}\ ) = 0.4004750 \times 10^{-2}$

$p_{11}\ (\bar{X}_o\ ) = 361.3024$ cycles/sec.

$(N_x)_{cr}\Big|_{\bar{X}_o} = 1941.111$ lb./in.

$w_{max}\ (\bar{X}_o\ ) = 0.3970621 \times 10^{-2}$ in.

(b) Total Number of plies = 12

Number of design variables, $N_2 = 6$

At starting design point :

$\bar{X}_o = 1.7 \times 10^{-2},\ 1.7 \times 10^{-2},\ 1.7 \times 10^{-2},\ 1.7 \times 10^{-2},$
$1.7 \times 10^{-2},\ 1.7 \times 10^{-2}$

$r_1 = 0.25 \times 10^{-5}$

$\phi\ (\bar{X}_o,\ r_1) = 0.5267113 \times 10^{-2}$

$F\ (\bar{X}_o) = 0.4228788 \times 10^{-2}$

$w_{max}(\bar{X}_o) = 0.3933030 \times 10^{-2}$ in.

$p_{11}(\bar{X}_o) = 381.51478$ cycles/sec.

$(N_x)_{cr}\Big|_{\bar{X}_o} = 2462.060$ lb./in.

At final design point :

$\bar{X}_{opt} = 2.00051 \times 10^{-2},\ 1.34472 \times 10^{-2},\ 1.76682 \times 10^{-2},$
$1.52163 \times 10^{-2},\ 1.62367 \times 10^{-2},\ 1.59800 \times 10^{-2}$

$r_4 = 0.25 \times 10^{-8}$

$\phi(\bar{X}_{opt}, r_4) = 0.4089727 \times 10^{-2}$

$F(\bar{X}_{opt}) = 0.4085897 \times 10^{-2}$

$w_{max}(\bar{X}_{opt}) = 0.3999654 \times 10^{-2}$ in.

$p_{11}(\bar{X}_{opt}) = 368.6234$ cycles/sec.

$(N_x)_{cr}\Big|_{\bar{X}_{opt}} = 2220.819$ lb./in.

(c) Total Number of plies = 16

Number of design variables, $N_3 = 8$

At starting design point :

$\bar{X}_o = 1.3 \times 10^{-2},\ 1.3 \times 10^{-2},\ 1.3 \times 10^{-2},\ 1.3 \times 10^{-2},$
$1.3 \times 10^{-2},\ 1.3 \times 10^{-2},\ 1.3 \times 10^{-2},\ 1.3 \times 10^{-2}$

$r_1 = 0.15 \times 10^{-5}$

$\phi(\bar{X}_o, r_1) = 0.4818080 \times 10^{-2}$

$F(\bar{X}_o) = 0.4311705 \times 10^{-2}$

$w_{max}(\bar{X}_o) = 0.3854437 \times 10^{-2}$ in.

$p_{11}(\bar{X}_o) = 388.99545$ cycles/sec.

$(N_x)_{cr}\Big|_{\bar{X}_o} = 2609.7453$ lb./in.

At final design point :

$$\bar{X}_{opt} = 1.69641 \times 10^{-2},\ 1.02909 \times 10^{-2},\ 1.50000 \times 10^{-2},\ 1.17626 \times 10^{-2},\ 1.18935 \times 10^{-2},\ 1.07873 \times 10^{-2},\ 1.12785 \times 10^{-2},\ 1.11559 \times 10^{-2}.$$

$r_4 = 0.15 \times 10^{-8}$

$\phi(\bar{X}_{opt}, r_4) = 0.4110638 \times 10^{-2}$

$F(\bar{X}_{opt}) = 0.4109910 \times 10^{-2}$

$w_{max}(\bar{X}_{opt}) = 0.3977044 \times 10^{-2}$ in.

$p_{11}(\bar{X}_{opt}) = 370.7898$ cycles/sec.

$(N_x)_{cr}\Big|_{\bar{X}_{opt}} = 2260.2045$ lb./in.

(d) Total Number of plies = 20

Number of design variables, $N_4 = 10$

At starting design point :

$$\bar{X}_o = 1.1 \times 10^{-2},\ 1.1 \times 10^{-2},\ 1.1 \times 10^{-2},\ 1.1 \times 10^{-2},\ 1.1 \times 10^{-2},\ 1.1 \times 10^{-2},\ 1.1 \times 10^{-2},\ 1.1 \times 10^{-2},\ 1.1 \times 10^{-2},\ 1.1 \times 10^{-2}$$

$r_1 = 0.25 \times 10^{-5}$

$\phi\ (\bar{X}_o, r_1) = 0.6856721 \times 10^{-2}$

$F\ (\bar{X}_o) = 0.4568749 \times 10^{-2}$

$w_{max}\ (\bar{X}_o) = 0.33172469 \times 10^{-2}$ in

$p_{11}\ (\bar{X}_o) = 412.18558$ cycles/sec.

$(N_x)_{cr}\Big|_{\bar{X}_o} = 3104.8663$ lb./in.

At final design point :

$$\bar{X}_{opt} = 1.30500 \times 10^{-2},\ 7.56153 \times 10^{-3},\ 1.16610 \times 10^{-2},\ 8.44384 \times 10^{-3},\ 1.07492 \times 10^{-2},\ 9.11366 \times 10^{-3},\ 1.01533 \times 10^{-2},\ 9.56735 \times 10^{-3},\ 9.80984 \times 10^{-3},\ 9.74539 \times 10^{-3}$$

$r_4 = 0.25 \times 10^{-8}$

$\phi\ (\bar{X}_{opt}, r_4) = 0.4142949 \times 10^{-2}$

$F\ (\bar{X}_{opt}) = 0.4139862 \times 10^{-2}$

$w_{max}\ (\bar{X}_{opt}) = 0.3980690 \times 10^{-2}$ in

$p_{11}\ (\bar{X}_{opt}) = 373.492$ cycles/sec.

$(N_x)_{cr}\Big|_{\bar{X}_{opt}} = 2309.982$ lb./in.

The relationship between the minimum weight and the number of plies is shown plotted in figure 7(e). It can be observed that the minimum weight varies with the change in number of plies for preassigned values of lower bounds on natural frequency and critical buckling load and

the upper bound on transverse deflection. The optimum number of plies seems to be 12 . The computer time used for each problem varied from nine minutes to thirtyfive minutes depending upon the number of design variables. The transverse deflection constraint was active at optimum point in all the cases.

CHAPTER IV

EXPERIMENTAL STUDY

The aim of this work is to find the natural frequencies of vibration of fibre reinforced orthotropic laminates at different fibre orientation angles because closed form solutions are not available for them.

4.1 EXPERIMENTAL CHARACTERIZATION OF A LAMINA

In order to find the natural frequencies of composite plates theoretically, five intrinsic macroscopic properties E_L, E_T, ν_{LT}, ν_{TL} and G_{LT} of a basic lamina or ply are required.

In theory, discrepancies exist between the various micro-mechanical expressions for predicting the fibre elastic constants. Therefore, it is necessary, certainly for the more important structures, that these quantities be based on a phenomenological approach and determined experimentally. A simple technique for accomplishing this is discussed in this section.

At least three test are required to determine the desired elastic properties in any direction θ from the following theoretical information :

$$\frac{1}{E_1} = \frac{\cos^4\theta}{E_L} + \left(\frac{1}{G_{LT}} - \frac{2\nu_{LT}}{E_L}\right)\sin^2\theta\cos^2\theta + \frac{\sin^4\theta}{E_T} \quad (115)$$

$$\frac{1}{E_2} = \frac{\sin^4\theta}{E_T} + \left(\frac{1}{G_{LT}} - \frac{2\nu_{LT}}{E_L}\right)\sin^2\theta\text{-}\cos^2\theta + \frac{\cos^4\theta}{E_T} \quad (116)$$

$$\frac{1}{G_{12}} = \frac{\cos^2 2\theta}{G_{LT}} + \left(\frac{1+\nu_{LT}}{E_L} + \frac{1+\nu_{TL}}{E_T}\right)\sin^2 2\theta \quad (117)$$

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} = \frac{\nu_{LT}}{E_L} - \frac{1}{4}\left(\frac{1+\nu_{LT}}{E_L} + \frac{1+\nu_{LT}}{E_T} - \frac{1}{G_{LT}}\right)\sin^2 2\theta \quad (118)$$

where the subscripts L and T denote the longitudinal and transverse directions respectively.

These tests can be simple tensile tests as shown in figure 10.

The test of figure 10(a) can be used to determine the elastic constants E_L and ν_{LT}. By measuring the load P, the cross-sectional area A, and the strains ε_1 and ε_2,

$$E_1 = \frac{\sigma_1}{\varepsilon_1} = \frac{P}{A\,\varepsilon_1} \quad \text{and} \quad \nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1} \quad (119)$$

can be established, since $\theta = 0^\circ$ for this case, equations (115) to (116) give,

$$\frac{1}{E_1} = \frac{1}{E_L}$$

or $$E_L = E_1 = \frac{P}{A\,\varepsilon_1} \quad (120)$$

and $$\frac{\nu_{12}}{E_1} = \frac{\nu_{12}}{E_L} = \frac{\nu_{LT}}{E_T}$$

or $$\nu_{LT} = \nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1} \qquad (121)$$

Since ε_1 is a tensile strain (positive) and ε_2 is a compressive strain (negative), ν_{LT} is positive in sign in equation (121).

In a similar manner, the test of figure 10(b) where $\theta = 90^{\circ}$, yields,

$$E_T = E_1 = \frac{P}{A\,\varepsilon_1} \qquad (122)$$

and $$\nu_{TL} = \nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1} \qquad (123)$$

A check on the accuracy of the measurements can be made at this point, since the equality, $E_L\,\nu_{TL} = E_T\,\nu_{LT}$, has to hold.

The final elastic constant G_{LT} can be determined by the test of figure 10(c). If the test is made such that $\theta = 45^{\circ}$ and

$$E_1 = \frac{P}{A\,\varepsilon_1}$$

is computed, the value of G_{LT} can be determined from equation (115) as:

$$\frac{1}{E_1} = \frac{1}{4}\left(\frac{1}{E_L} + \frac{1}{E_T} + \frac{1}{G_{LT}} - \frac{2\,\nu_{LT}}{E_L}\right)$$

i.e. $$\frac{1}{G_{LT}} = \frac{4}{E_1} - \frac{1}{E_L} - \frac{1}{E_T} + \frac{2\,\nu_{LT}}{E_L}$$

4.2 EXPERIMENTAL SET-UP FOR MEASURING NATURAL FREQUENCIES OF A LAMINATED PLATE

The natural frequencies of vibration for simply supported square plates were determined experimentally. To simulate supported boundary conditions, initially balls of 1/4" diameter were used. These balls were placed at a distance of $1\frac{1}{2}$" along the edge on both sides of the plate. The lower set of balls was placed in holes in an angle structure fixed to the ground and the upper set of balls was held in the holes of m.s. flats with the help of grease. To varify the boundary conditions simulated, the natural frequencies of an aluminium plate were found on the test set-up. A tightening pressure of about 1/2 lb.-in., with the help of a torque wrench, was applied for the aluminium plate and the results obtained were found not to be matching with the theoretically calculated values. Also the reproducibility of the results could not be achieved.

Hence another attempt was made by replacing the balls by a 0.2 in. diameter bright bar, as shown in figure 11 . The tightening pressure applied was just sufficient to make the rod just touch the test plate. This was verified by making sure that a razor blade does not slip in between the rod and the plate on either side. With this arrangement, an excellent agreement between the experimental and the theoretical values, with a very good reproducibility of the results for aluminium plate, was obtained.

The test plates were excited indirectly with accoustic excitation from a loud speaker. A sinusoidal input from a RC oscillator through a power amplifier was fed to the loud-speaker which imparted the excitation energy to the plate through sinusoidal sound-waves. An electronic counter was used to note the frequency of the input signal. The block-diagram representation of the set-up is given in figure 10.

A small piece of magnetic material (razor blade) was fixed to the composite plate with the help of cellulose tape and a contactless pick-up was used to find the frequency of vibration of the plate which was found to be matching with the input signal at resonance.

The mode shapes were detected visually by spraying very fine sand on the plate surface. The final adjustment of the input frequency was made by displaying the signal of the contactless pick-up on an oscilloscope to ensure that the signal has a maximum amplitude.

4.3 TEST SPECIMEN

The vibration tests (for natural frequencies) were conducted on uni-and bi-directionally reinforced composite plates. In bidirectional plates, five layers of the E-glass fabric and a polyester laminate cured with

MEK peroxide and cobalt octate were used but the orientation of all of them was same with respect to the plate axis. The orientations tested were 0°, 15°, 30° and 45°, since the angles between 45° and 90° will be just the repetition of these only. The plates were terminated to a size of 9.2" x 9.2" and the thickness was approximately 0.034".

Three unidirectionally reinforced plates were fabricated from E-glass roving and CY 230 epoxy resin (cured with 10% HY 951 hardener). These plates were almost identical in thickness and in volume concentration of reinforcement. They were cut to the same size and glued together using the same resin as the bonding agent; the 90° plate was kept in the middle layer. The cured plate was trimmed to 9.2" x 9.2" size and the average thickness of the plate was 0.1257". The plate was so thick that only the fundamental frequency could be excited.

4.4 EXPERIMENTAL RESULTS

In this section the results obtained experimentally are listed. Since the fibre reinforced composites are not readily available commercially, a limited number of tests were performed.

4.4.1 Elastic Constants

The tests as described in article (4.1) were carried out on an Instron Testing Machine. The strains were measured with the help of strain gauges mounted on the test specimen. The load vs strain is shown plotted in figures 8 and 8(a) for uni - and bi-directionally reinforced specimens respectively and the elastic constants found are given below.

For unidirectional specimen:

$E_L = 2.137 \times 10^6$ psi; $E_T = 0.301 \times 10^6$ psi;

$G_{LT} = 0.257 \times 10^5$ psi; $\nu_{LT} = 0.28$ and

$\rho = 0.00439$ lb/in.^{3}

For bidirectional specimen :

$E_L = 15.27 \times 10^6$ psi; $E_T = 14.5 \times 10^6$ psi;

$G_{LT} = 0.189 \times 10^5$; $\nu_{LT} = 0.1475$ and

$\rho = 0.002083$ lb/in.^{3}

4.4.2 Natural Frequencies

The natural frequencies of vibration of bi-directionally reinforced composite plates at different orientation angles are given in Tables (4.1) to (4.4). The size of all the plates within the simply supported edges was 9.015" x 9.017" and the overall plate dimensions were approximately 9.2" x 9.2".

For 0° plate : Thickness of the plate = 0.0371 inches

TABLE 4.1 Natural Frequencies (cycles/sec.)

n ↓ \ m →	0	2	4
0	69.1	273.0	690.7
2	281.7	-	-
4	729.0	-	-

The mode shapes of vibration are shown in Figure 13.

For 15° plate : Thickness of the plate = 0.0343 inch

TABLE 4.2 Natural Frequencies (cycles/sec.)

n ↓ \ m →	0	2	4
0	56.4	246.7	635.2
2	261.5	457.4	-
4	670.3	-	-

The mode shapes of vibration are shown in Figure 14.

For 30° plate : Thickness of the plate = 0.0341 inch

TABLE 4.3 Natural frequencies (cycles/sec.)

n ↓ \ m →	0	2	4
0	59.5	235.0	593.8
2	250.9	463.4	-
4	615.3	-	-

The mode shapes of vibration are shown in Figure 15.

For 45° plate : Thickness of the plate = 0.0342 inch.

TABLE 4.4 Natural frequencies (cycles/sec.)

n ↓ \ m →	0	1	2
0	62.8	-	249.2
1	-	224.0	-
2	258.0	-	280.0

The mode shapes of vibration are shown in Figure 16.

The unidirectionally reinforced laminated plate was very thick and the excitation energy required for getting higher modes of vibration was much more than the energy of the available source of excitation.

The fundamental frequency of the plate was found to be 157.5 cycles/sec.

Theoretical fundamental frequency of the plate for varying ply thickness with thickness of all three plies same, is shown plotted in Figure 17.

4.5 DISCUSSION OF EXPERIMENTAL RESULTS

4.5.1 Nodal Patterns of Orthotropic Plates

The nodal patterns of the laminated plates which behave as orthotropic plates may be divided into three classes; one consisting of parallel nodal lines, one involving non-parallel nodal lines and one having compounded normal modes.

For the simply supported boundary conditions, the mode shapes are comprised only of the rectangular antinodal regions with nodal lines running parallel to the plate edges. The parallel nodal lines attribute to the existance of (exact) mode shapes for which a separation into a sine wave in the x - direction and a waveform in the y - direction independent of x is possible.

In the steady-state vibration considered, certain theoretical modes were not detected. These modes have an odd number of nodal lines in the x - and y - directions. This phenomenon occurs when two opposite

edges have similar edge conditions and the plate cannot vibrate in the modes having odd number of nodal lines parallel to these edges under the imposed displacement excitation. This also can be explained physically by the method of excitation, and the symmetry of a plate geometry, elastic properties and boundary conditions about the line bisecting the distance between these two edges. The equivalent uniform external force induced by the applied imposed displacement is also symmetric. Since the mode shapes depend exclusively on these factors, the plate can vibrate only in the symmetric mode shapes which have only even number of nodal lines.

4.5.2 Nodal Patterns of Anisotropic Plates

No attempt was made to predict the nodal patterns of anisotropic plates. The effect of anisotropy (D_{13} and D_{23}) causes an inclination of straight, otherwise parallel, lines or a skewing and bending of patterns. This was predominately clear for 45° orientation.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

5.1 CONCLUSIONS

1. The present study shows the feasibility of obtaining theoretical optimum parameters of laminated composite beams and plates subjected to multiple behaviour constraints.

2. The curves plotted in figure 7 for the five theoretical optimization problems show that the frequency of vibration of laminated orthotropic plates varies linearly with a change in the weight of the plate i.e. the thickness of the plate, and is fairly insensitive to the change in number of plies. However, the critical buckling load and the transverse deflection depend upon the number of plies. The minimum weight of the plate for a constant buckling load decrease first and then increases with an increase in the number of plies. A similar trend is observed in the case of bound on the transverse deflection.

3. The experimental study indicates that the

square orthotropic plate increases as the fibre orientation angle varies from 0° to 45°. No definite pattern has been observed for the higher natural frequencies.

5.2 RECOMMENDATIONS FOR FUTURE WORK

1. The optimization of laminated composite plates can be attempted by taking the fibre-orientation angles of the lamina as design variables. This requires the development of analysis procedures for predicting the natural frequencies and critical buckling loads of such plates.

2. The behaviour of the critical buckling load and the transverse deflection can be predicted theoretically for constant number of plies and varying values of the weight of the plate.

3. The optimization of composite shell structures can be done in a similar manner.

4. It is recommended to employ another means of excitation so that all modes and various compounded nodal patterns consisting of same constituent modes can be detected. This can be achieved through the use of point exciters, which are capable of exciting the plate at any location.

5. In order to verify the reduced effective flexural stiffnesses, a further experimental study should be made by testing a series of unbalanced laminates and by measuring the plate surface strains or stresses.

REFERENCES

1. E. Reissner, and Y. Stavsky, "Bending and Stretching of Certain Types of Heterogeneous Aeolotropic Elastic Plates," J. Applied Mechanics, Vol. 28 (1961), p. 402.

2. S.B. Dong, R.B. Matthiesen, K.S. Pister, and R.L. Taylor, "Analysis of Structural Laminates," Air Force Report ARL - 76 (1961).

3. J.M. Whitney and A.W. Leissa, "Analysis of Heterogeneous Anisotropic Plates," J. Applied Mechanics, Vol. 36 (1969), p. 261.

4. J.M. Whitney, "Bending-extensional Coupling in Laminated Plates Under Transverse Loading," J. Composite Materials, Vol. 3 (Jan. 1969), p. 20.

5. Yuichi Hikami, "Transverse Vibrations of Laminated Orthotropic Plates," Air Force Materials Laboratory, Report, Wright-Patterson Air Force Base, Ohio, Report No. 32 (April 1969).

6. John F. Mandell, "An Experimental Investigation of the Buckling of Anisotropic Fiber Reinforced Plastic Plates," Report, Air Force Materials Laboratory, Wright-Patterson Air Force Base, Ohio, AFML-TR-68-281 (Oct., 1968).

7. Lee R. Calcote, "The Analysis of Laminated Composite Structures," New York, Van Nostrand Reinhold Company, 1969.

8. L.A. Schmit and B. Farshi, "Optimal Laminate Design for Strength and Stiffness," International J. for Numerical Methods in Engineering, Vol. 7 (1973), p. 519.

9. R. Foye, "Advanced Design Concepts for Advanced Composite Air Frames," Report, Vol. I - II, Air Force Materials Laboratory, Wright-Patterson Air Force Base, Ohio, AFML-TR-68-91 (1968).

10. M.E. Waddoups, "Structural Airframe Application of Advanced Composite Materials - Analytical Methods," Report, Vol. VI, Air Force Materials Laboratory, Wright-Patterson Air Force Base, Ohio, AFML-TR-69-101 (1969).

11. M.E. Waddoups, L.A. McCullers, F.O. Olsen and J.E. Ashton, "Structural Synthesis of Anisotropic Plates". Paper presented at the AIAA/ASME 11th Struct., Struct. Dyn. and Matls Conf., Denver, Colorado (1970).

12. T.P. Kicher and T.L. Chao, "Minimum Weight Design of Stiffened Fibre Composite Cylinders," J. Aircraft, Vol. 8, (1971) p. 562.

13. Zvi Hashin, "Plane Anisotropic Beams", A.S.M.E. J. Appl. Mech., Vol. 34, (June, 1967), p. 257.

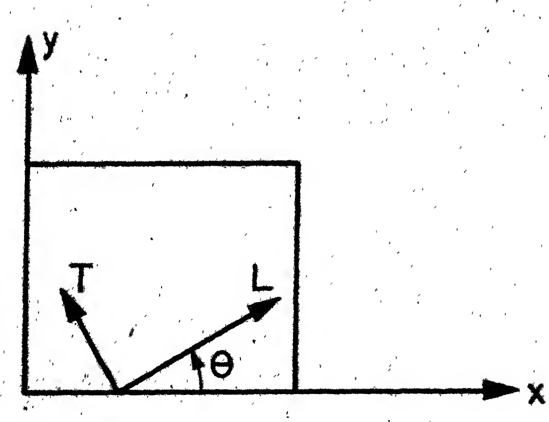


Fig. I. Ply Orientation Relative To Structure Axes

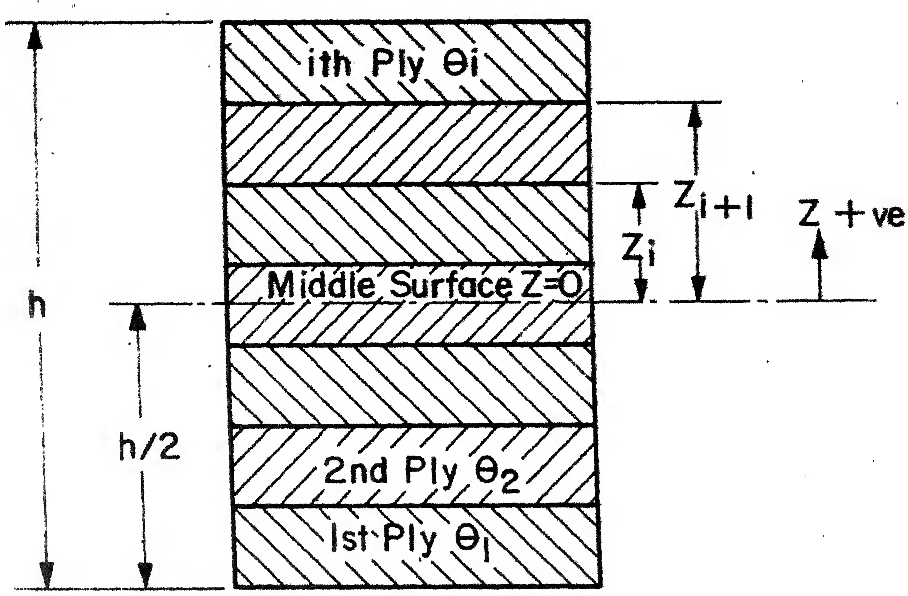


Fig. 2 Laminated Plate Cross-Section

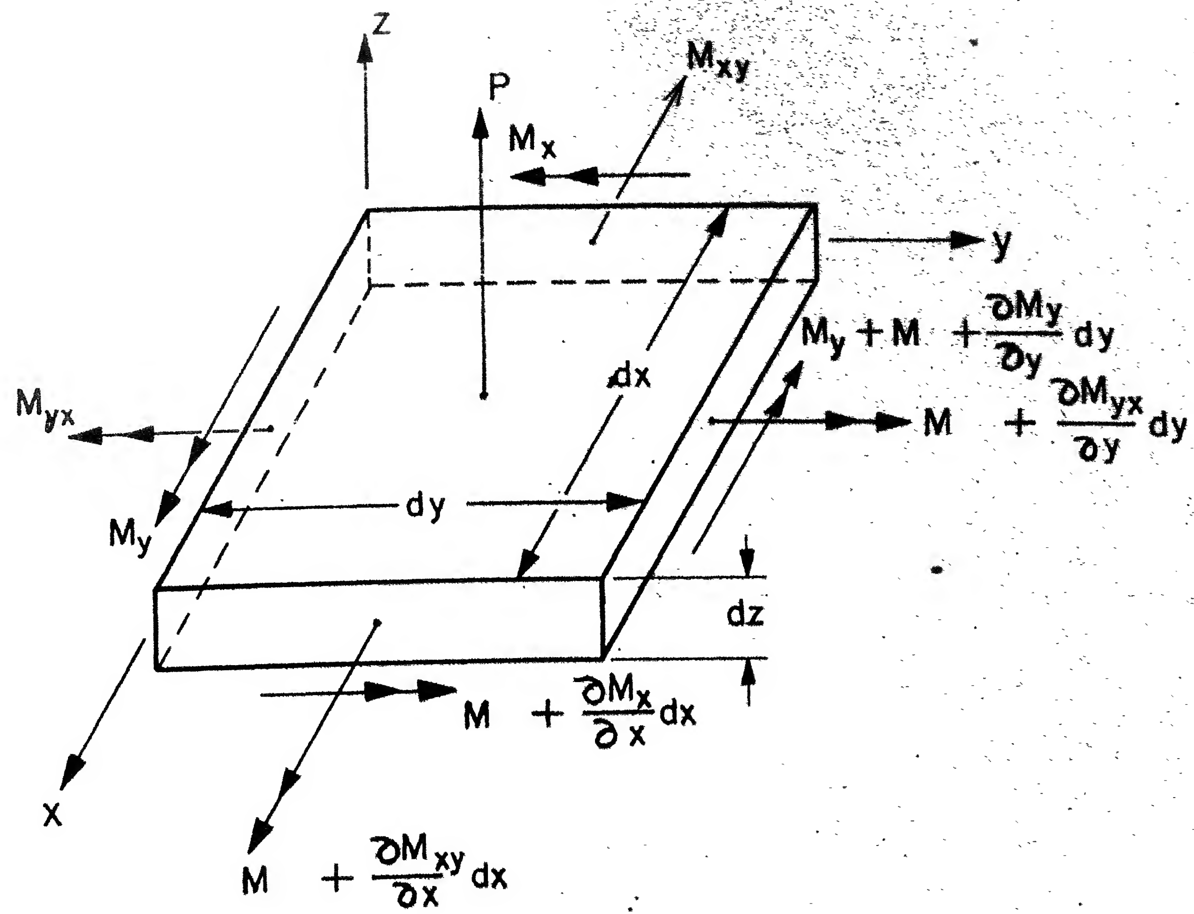


Fig.5 Laminated Element Subjected To Moments And Forces

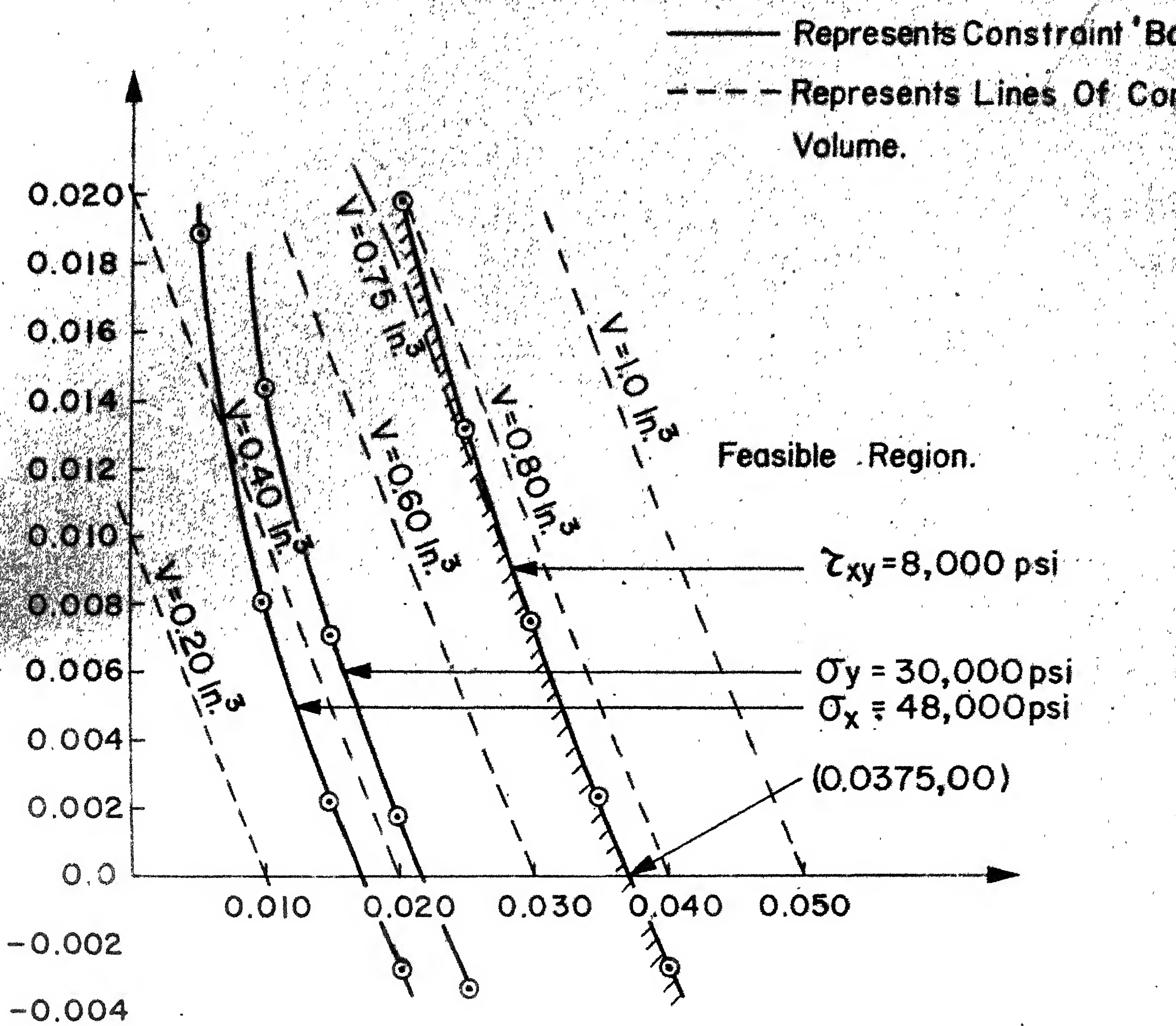
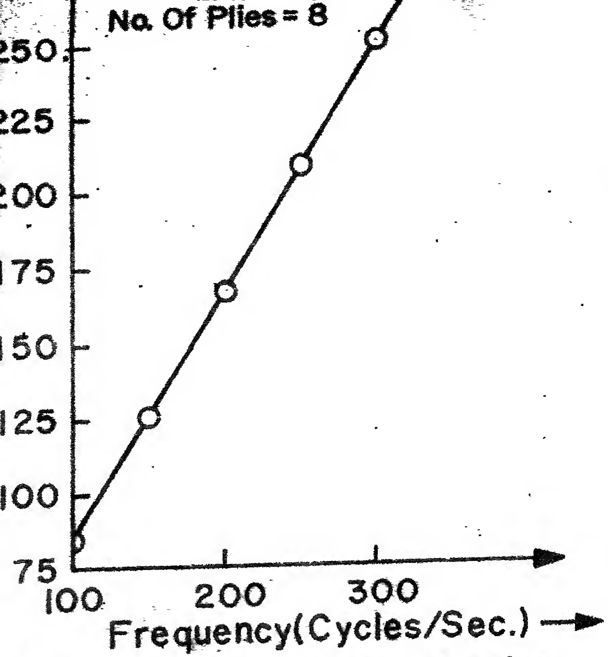
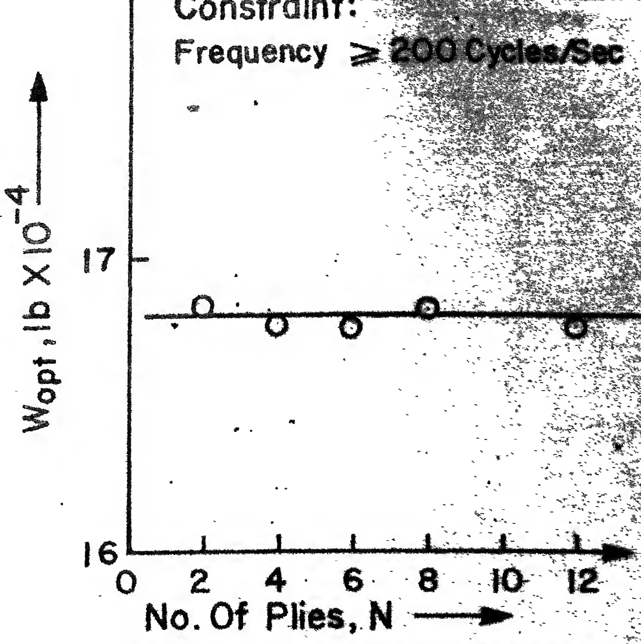


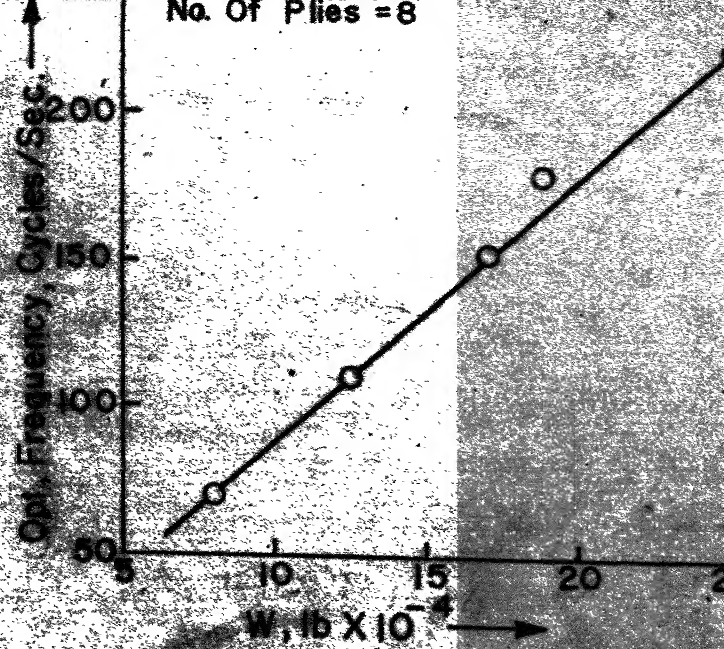
Fig. 6 Graphical Solution Of Laminated Beams With Constraints On Stresses.



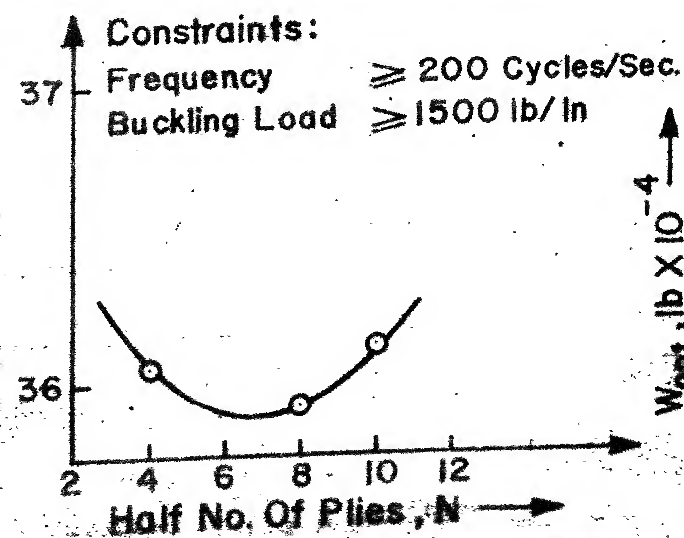
(a) For Constant No. Of Plies



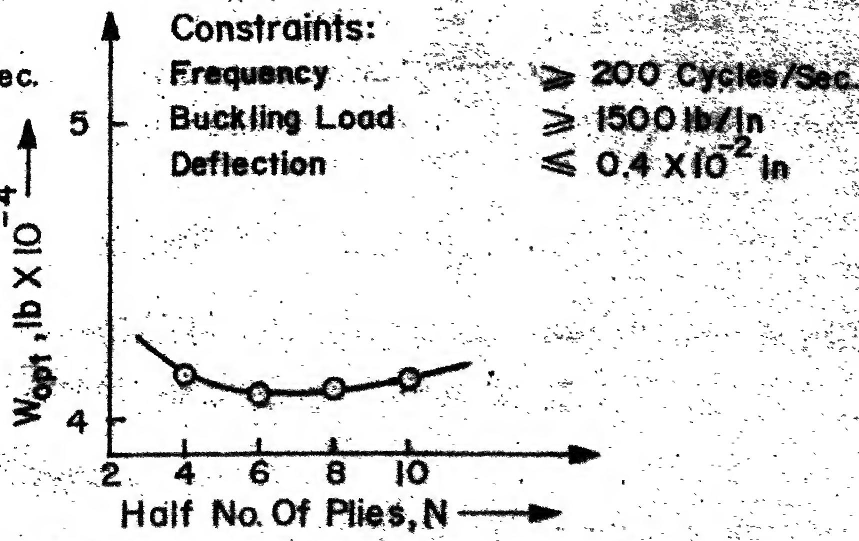
(b) For Constant Frequency



(c) For Constant No. Of Plies



(d) For Constant Frequency And Buckling Load



(e) For Constant Frequency, Buckling Load, And Deflection.

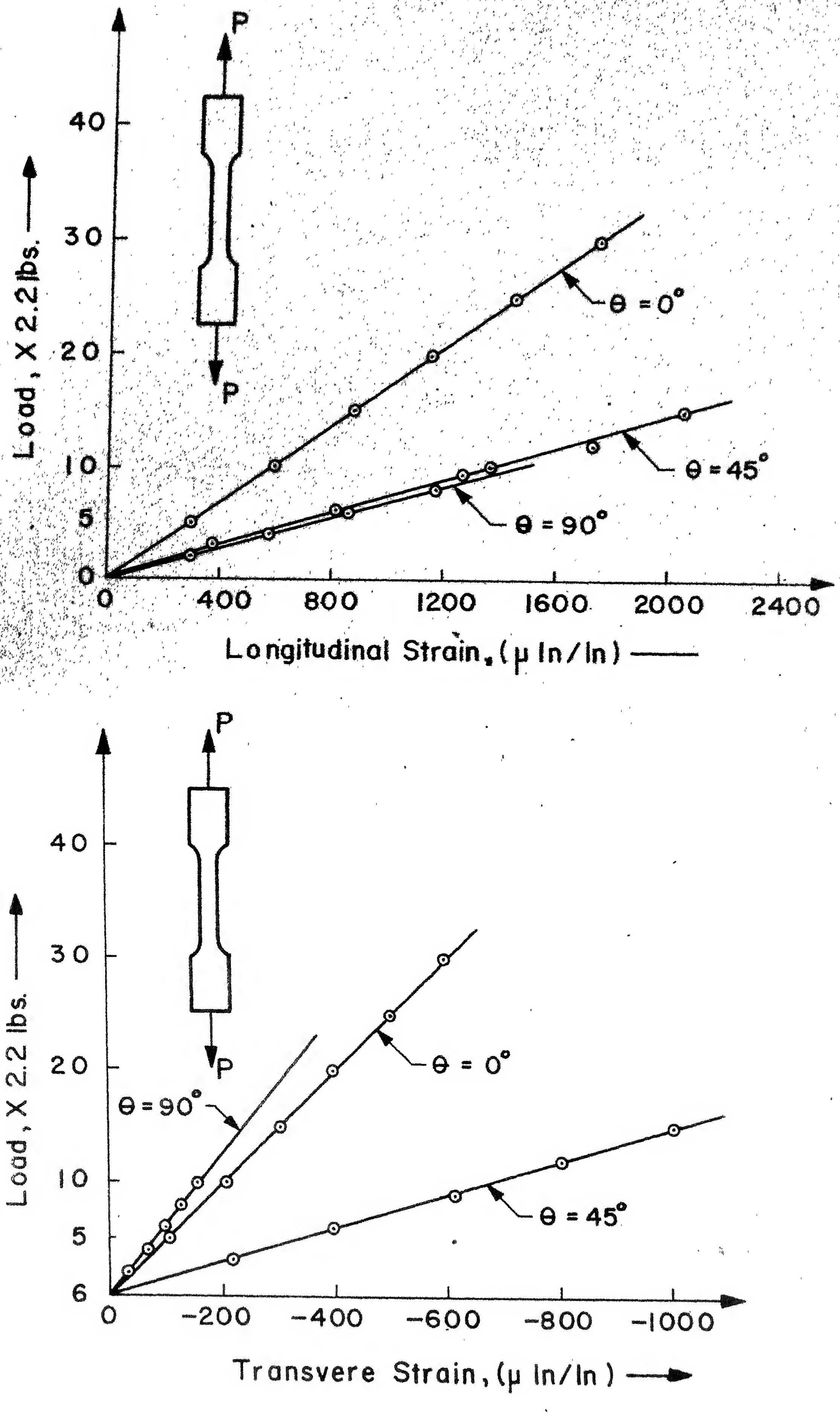


Fig.8 Load-Strain Curves For The Unidirectionally

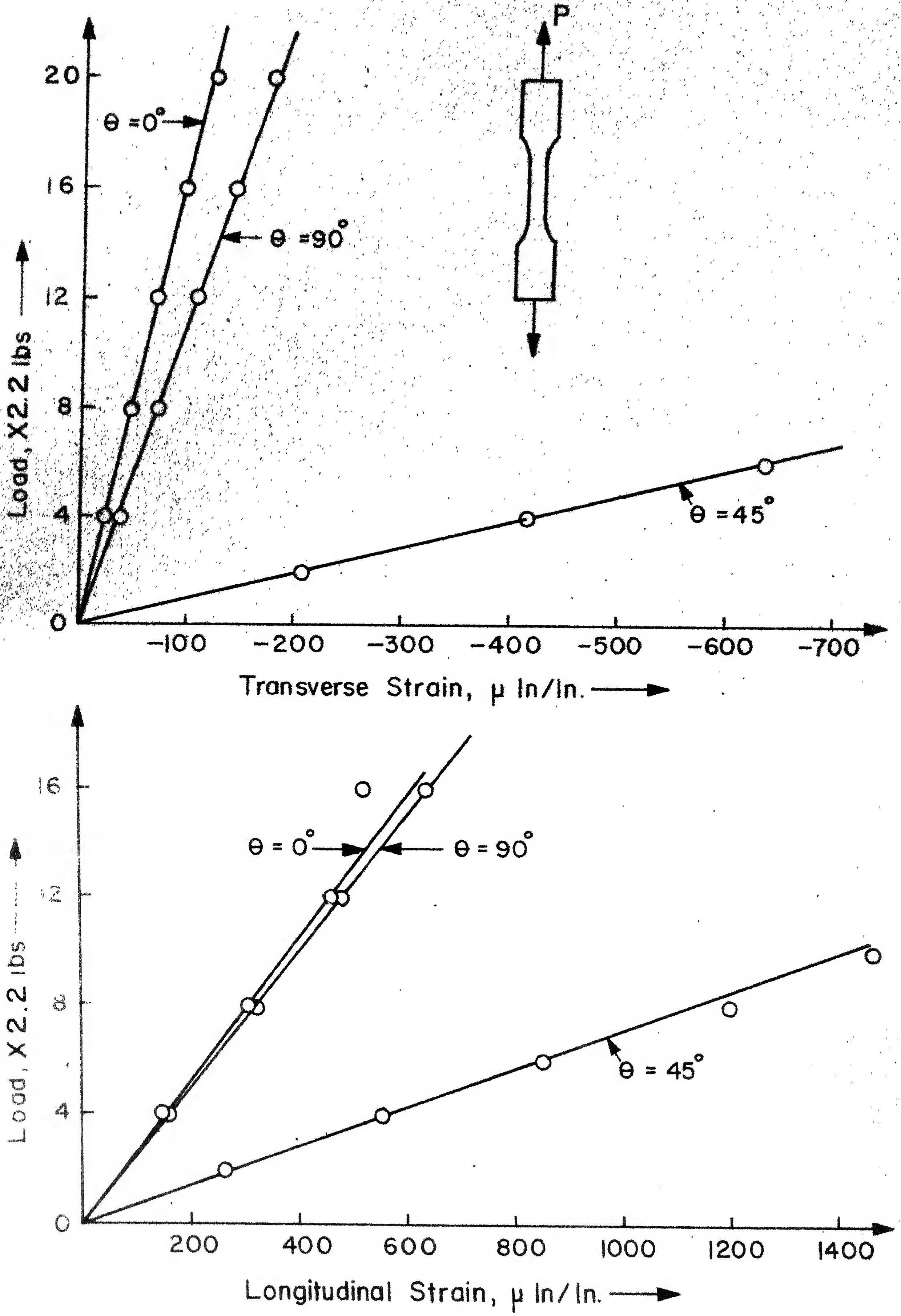


Fig. 8 (a) Load - Strain Curves For The Bidirectionally Reinforced Plate

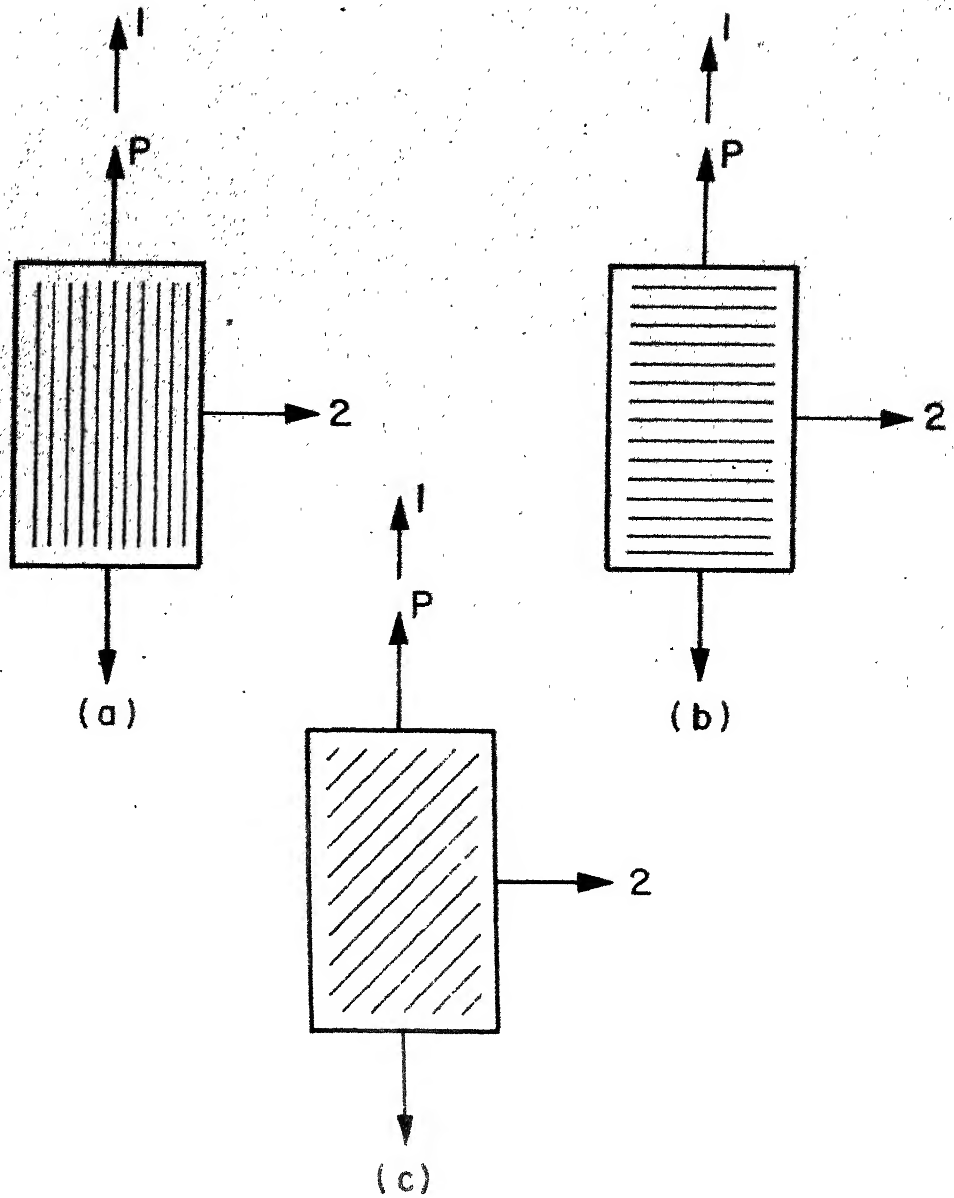


Fig. 9 Tensile Test For Finding Material Constan
Of a Laminate.

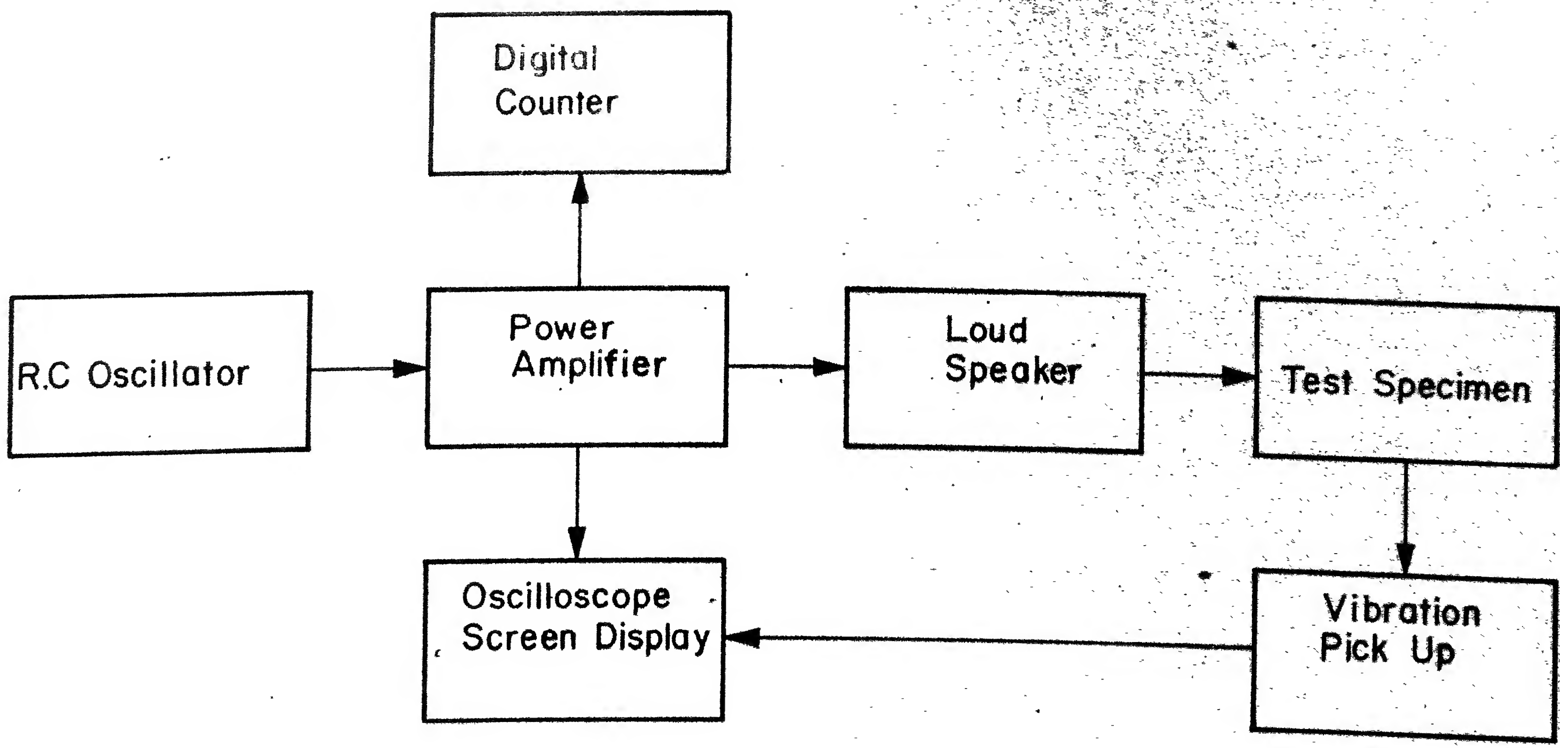


Fig. 10 Block Diagram Of The Experimental Set Up

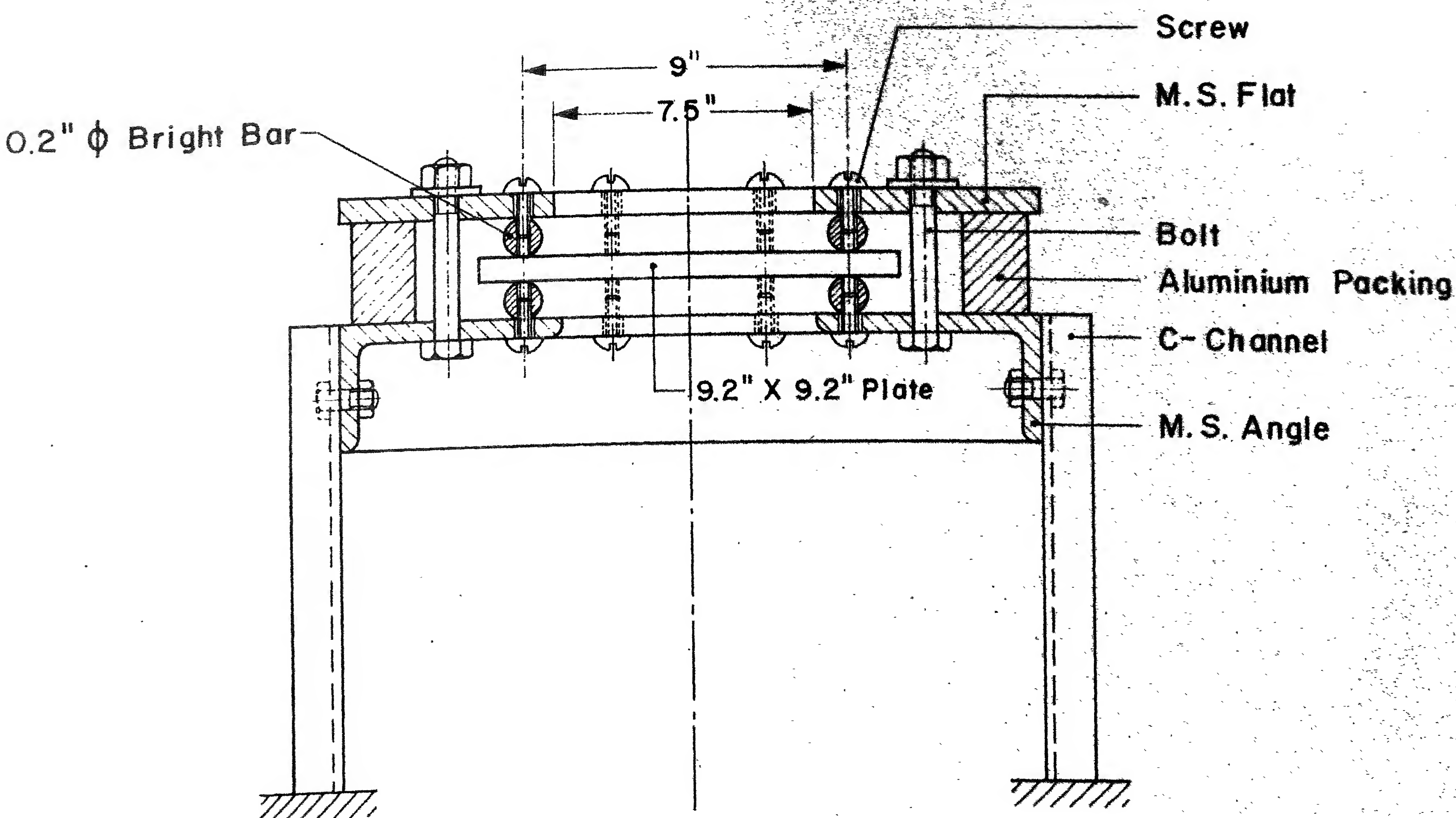
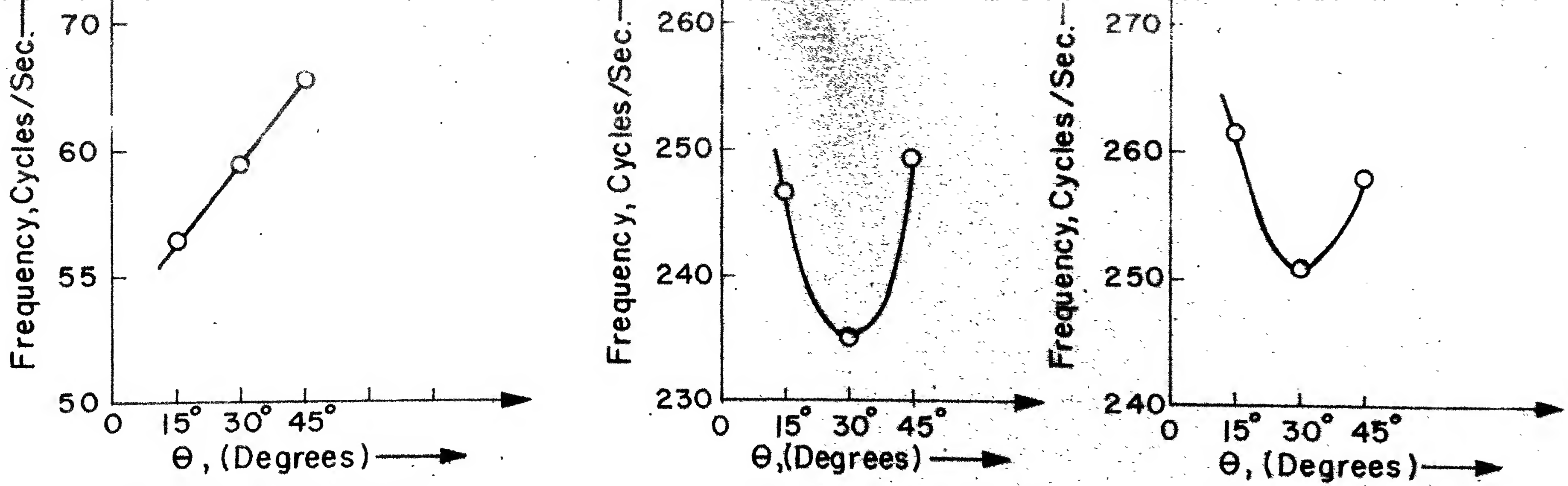
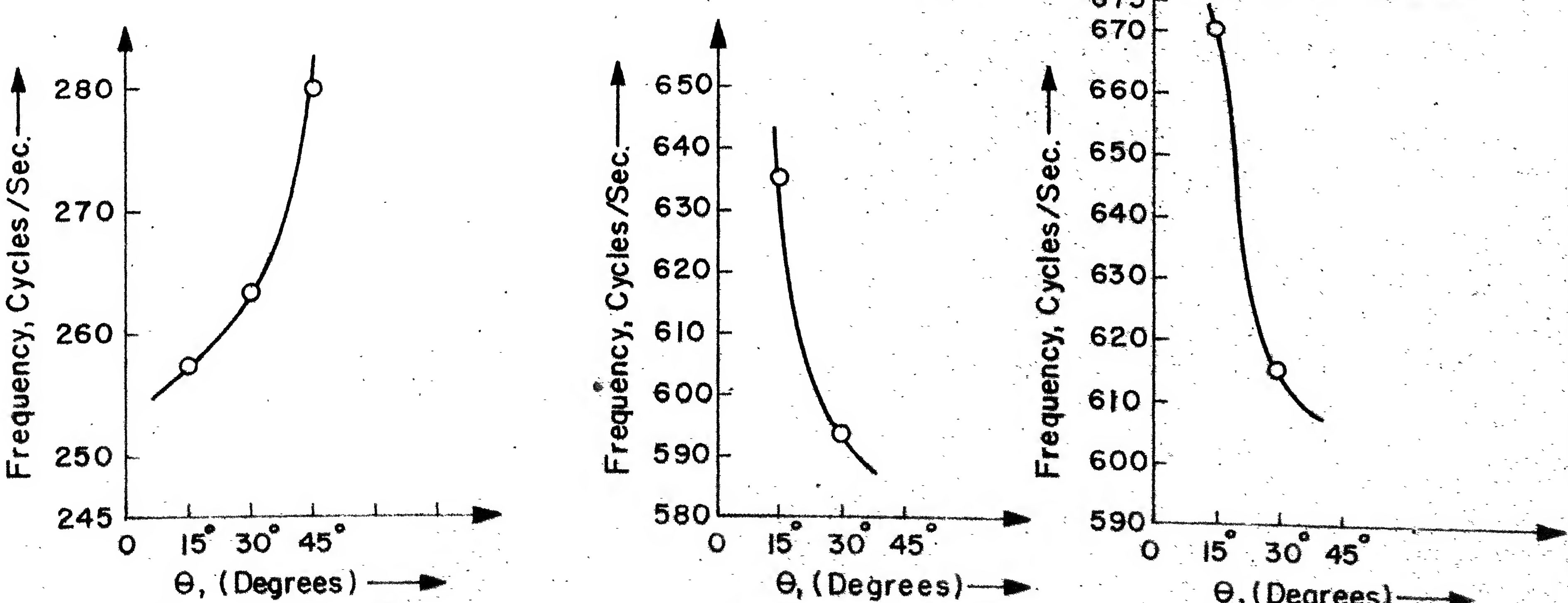


Fig. II Cross-Section Of Test Fixture
(A Schematic Representation)



(a) For n = 0, m = 0 (b) For n = 2, m = 0 (c) For n = 0, m = 2



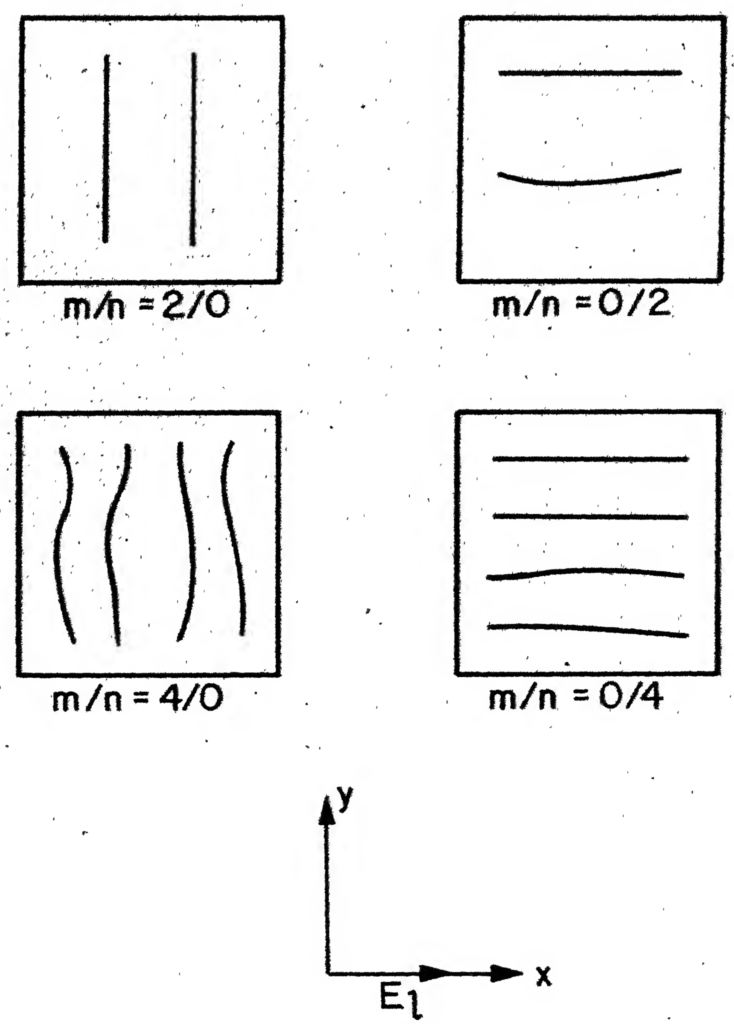


Fig. 13 Mode Shapes For $\Theta = 0°$ Plate

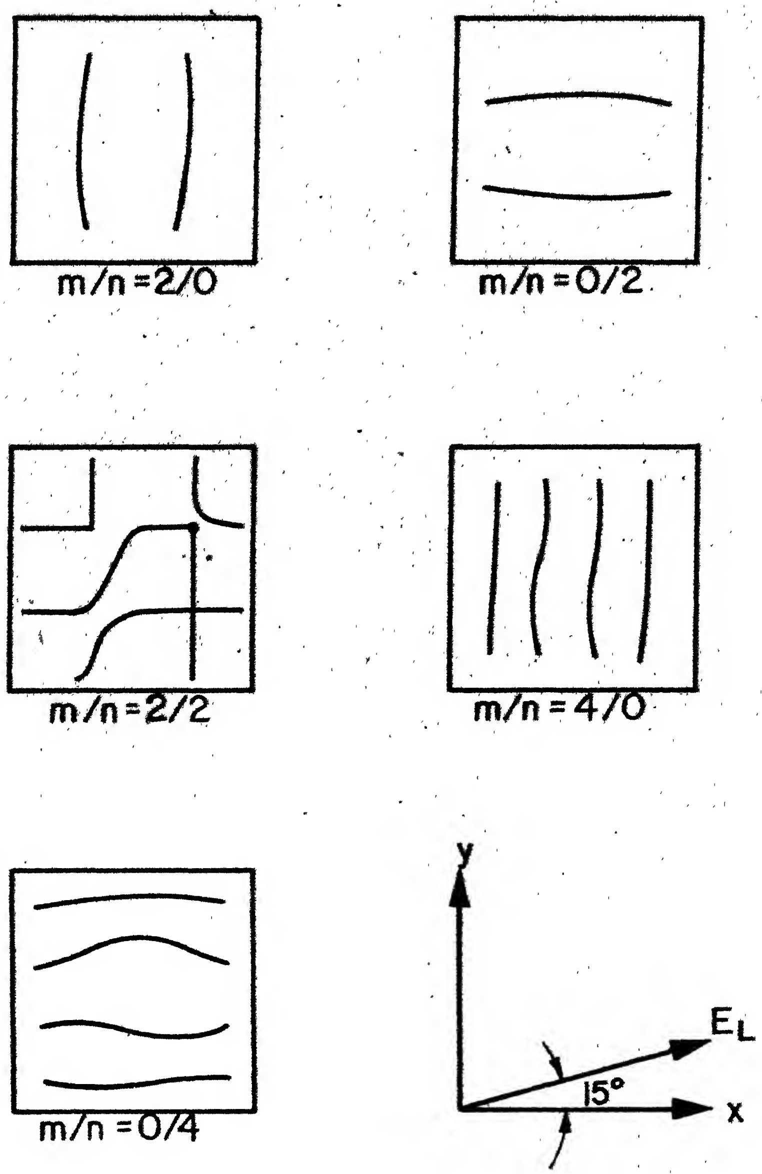


Fig. 14 Mode Shapes For $\Theta = 15°$ Plate.

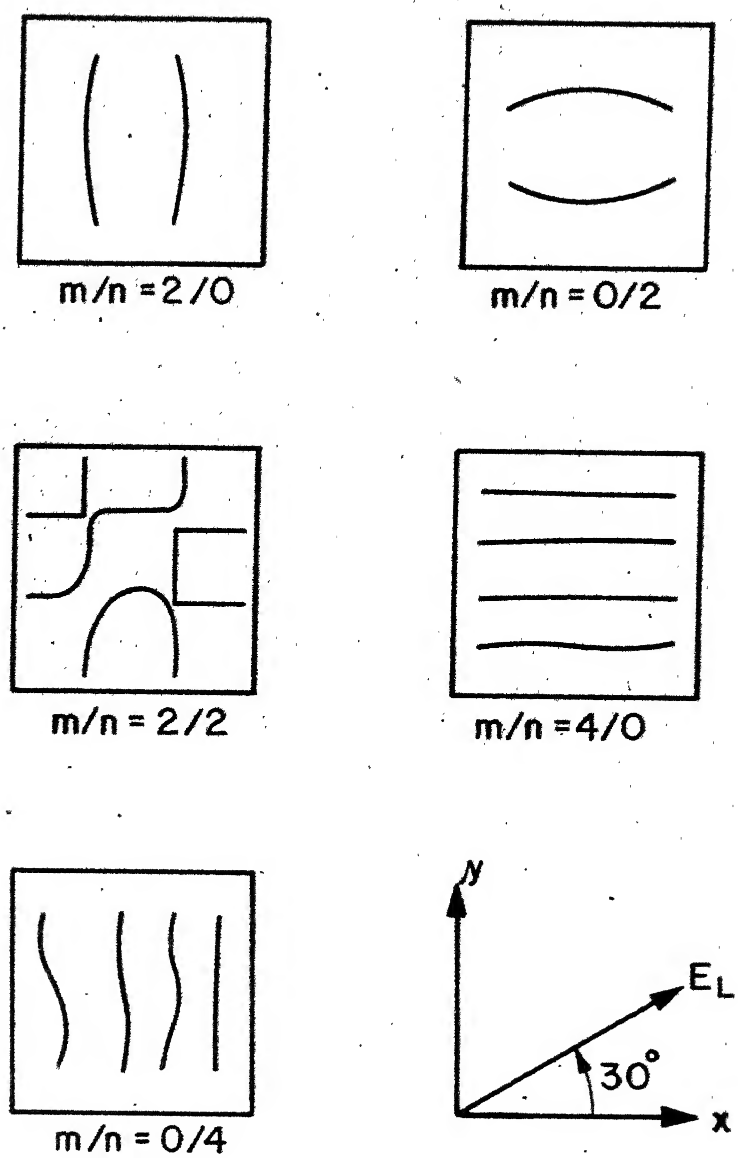


Fig. 15 Mode Shapes For $\theta = 30^\circ$ Plate

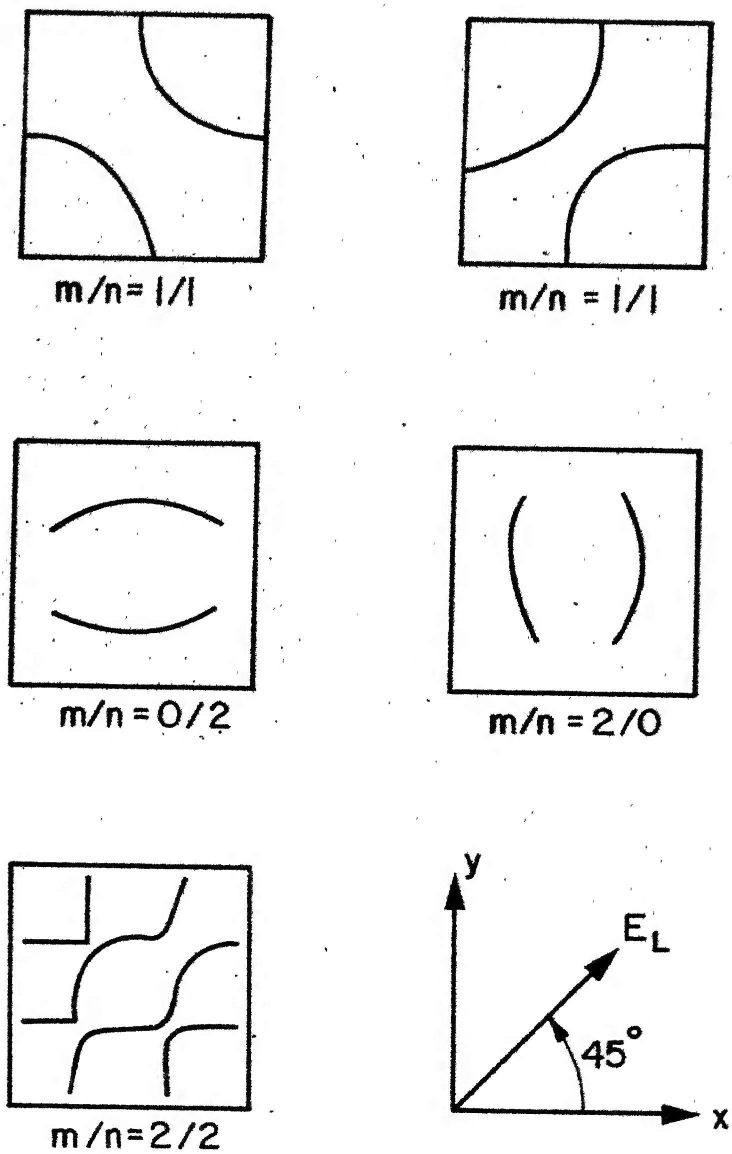


Fig. 16 Mode Shapes For $\Theta = 45^\circ$ Plate

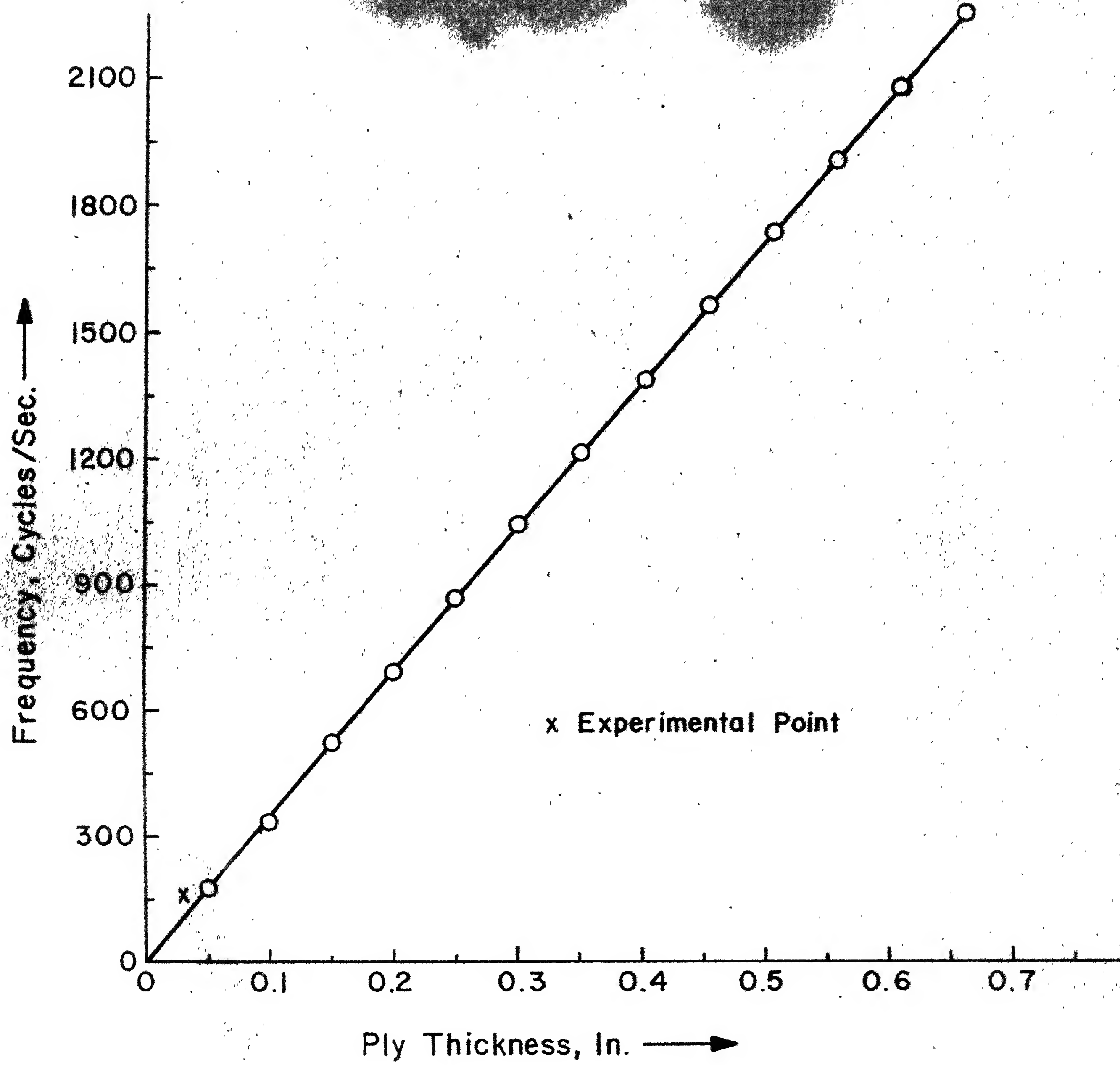


Fig. 17 Theoretical Variation Of Frequency With Thickness.

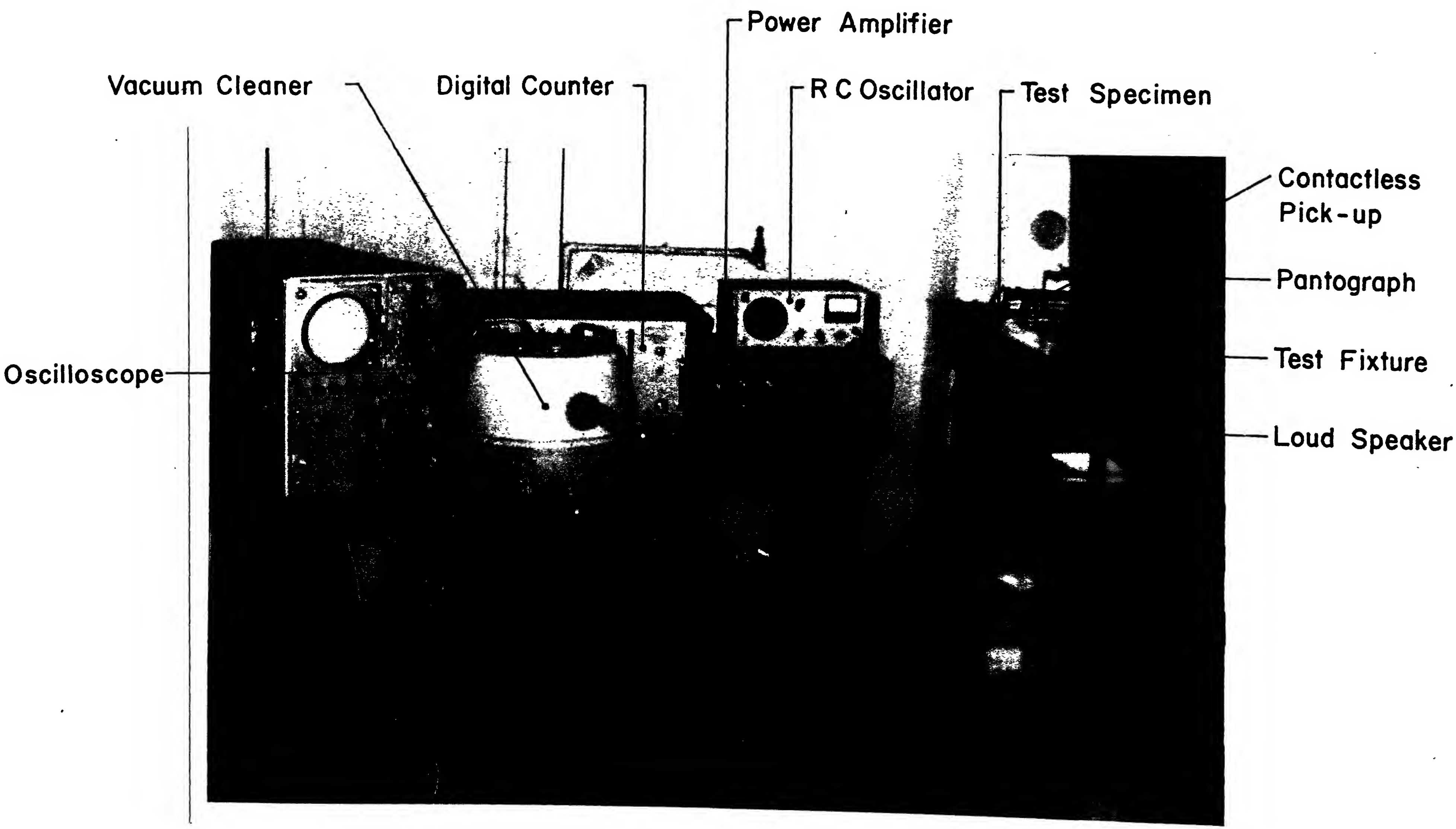


Fig. 18 Photograph Of The Experimental Set-up